

Spectral Statistics of Sparse Random Graphs

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Abstract. We survey recent progress on the spectral statistics of sparse random graphs, focusing on the Erdős–Rényi graph model and random d -regular graphs. Adjacency matrices of sparse random graphs form a canonical random-matrix ensemble with sparsity and dependent entries: most entries vanish, and degree constraints induce long-range correlations absent from Wigner models. We first present bulk, edge, and eigenvector universality results for these models. In the fixed-degree regime, edge universality for random d -regular graphs implies that asymptotically about 69% of d -regular graphs are Ramanujan. We then review the classical three-step strategy for proving universality of Wigner matrices and explain the additional ingredients required in the sparse setting, highlighting in particular the role of loop equations.

1 Introduction. Let $\mathcal{G}$ be a simple, undirected random graph on the vertex set $\llbracket N \rrbracket := \{1, \dots, N\}$. Its adjacency matrix $A = [A_{ij}]_{1 \leq i, j \leq N} \in \{0, 1\}^{N \times N}$ is defined by $A_{ij} = 1$ if and only if i and j are adjacent. By *spectral statistics* of $\mathcal{G}$, we mean quantitative properties of the eigenvalues and eigenvectors of A : for example, the sizes of the largest and second-largest eigenvalues, local eigenvalue spacings, and the distribution and delocalization of eigenvector entries. This survey reviews recent progress on the spectral statistics of *sparse* random graphs, with an emphasis on the Erdős–Rényi model $\mathbb{G}(N, p_N)$ and random d -regular graphs. In the rest of this introduction, we will introduce these graph models, and discuss some motivations and applications.

1.1 Erdős–Rényi graphs. The Erdős–Rényi graph model $\mathbb{G}(N, p_N)$ is the random graph on vertex set $\llbracket N \rrbracket$ in which each edge is included independently with probability $0 \leq p_N \leq 1$. In the *sparse* regime we allow $p_N \rightarrow 0$ as $N \rightarrow \infty$ (e.g., $p_N = \Theta(1/N^\alpha)$ for $0 \leq \alpha \leq 1$). Its adjacency matrix $A = [A_{ij}]_{1 \leq i, j \leq N}$ is symmetric with zero diagonal, and for $i < j$ the entries are independent with $A_{ij} = A_{ji} \sim \text{Bernoulli}(p_N)$; that is,

$$(1.1) \quad \mathbb{P}(A_{ij} = 1) = p_N, \quad \mathbb{P}(A_{ij} = 0) = 1 - p_N.$$

The expected degree of a vertex is $(N - 1)p_N \asymp Np_N$. We introduce the normalized adjacency matrix H and its centered version H° :

$$H := \frac{A}{\sqrt{N p_N (1 - p_N)}}, \quad H^\circ := H - \mathbb{E}[H] = \frac{A - \mathbb{E}[A]}{\sqrt{N p_N (1 - p_N)}}.$$

For $i \neq j$, the off-diagonal entries satisfy $\text{Var}(H_{ij}) = \text{Var}(H_{ij}^\circ) = 1/N$, and $H_{ii} = 0$.

1.2 Random d -regular graphs. A graph on N vertices is d -regular if every vertex has degree d . Necessarily $0 \leq d \leq N - 1$ and dN is even. In this article we mainly consider the *uniform model* $\mathcal{G}_{N,d}$: sample uniformly from the finite set of all *simple* (no loops or multiple edges) d -regular graphs on N vertices. Besides the uniform model, several other models, such as the *permutation model* and the *matching model* are considered in the literature. These models yield a d -regular *multigraph* with possible self-loops and multiple edges; conditioning on simplicity gives the uniform model $\mathcal{G}_{N,d}$. In this article we mainly discuss the uniform model, but many results also hold for other models.

We can identify a d -regular graph with its adjacency matrix A . For d -regular graphs, it is immediate that A has a trivial eigenvalue d with associated eigenvector $\mathbf{e} = (1, 1, \dots, 1)^\top$. Moreover, by the Perron–Frobenius theorem, all other eigenvalues are bounded in absolute value by d . For convenience, we shall consider the normalized adjacency matrix

$$(1.2) \quad H := A/\sqrt{d-1}.$$

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1.3 Motivation and Applications In this section we outline some motivations and applications that benefit from a precise understanding of spectral statistics of sparse random graphs.

Network hypothesis testing. Spectral statistics of random graphs provide powerful test statistics for network hypothesis testing. They provide tests that distinguish a null Erdős–Rényi model from structured alternatives—such as planted communities, denser subgraphs, or low-rank perturbations—via eigenvalue outliers and (de)localization of eigenvectors. For example, leading eigenvalue/eigenvector behavior in the adjacency matrices reveal planted cliques or other rank-one signals [5].

Expanders. *Expander graphs* are sparse yet highly connected networks, central to robust network design, error-correcting codes, and pseudorandomness. The seminal works of Alon–Milman [6] and Alon [4] relate a regular graph’s expansion properties to the magnitude of its *second-largest eigenvalue* (equivalently, to its spectral gap). For any fixed $d \geq 3$, a random d -regular graph is, with high probability, a strong expander—typically exhibiting near-optimal spectral expansion; see the excellent survey on expander graphs [87] for more details.

Ramanujan graphs. Lubotzky, Phillips, and Sarnak [112] introduced the concept of a *Ramanujan graph*—namely, a graph for which, aside from the “trivial” eigenvalues λ_1 and, if the graph is bipartite, $\lambda_N = -\lambda_1$, all others lie in the interval $[-2, 2]$. By the Alon–Boppana bound [123], this is the smallest possible interval that can contain these eigenvalues. Lubotzky, Phillips, and Sarnak [112] and Margulis [117] independently constructed infinite families of Ramanujan graphs of fixed degree $d \geq 3$ with $d - 1$ prime. A few other algebraic constructions have followed [97, 122, 124, 51], all of which apply only when $d - 1$ is prime or a prime power.

Establishing the existence of infinite families of Ramanujan graphs for every degree $d \geq 3$ has been a major open problem. Marcus, Spielman, and Srivastava fully solved the existence problem for *bipartite* graphs, showing that for any $d \geq 3$ and N there exists a bipartite d -regular Ramanujan graph on N vertices [115, 116]. Their methods also led to deterministic polynomial-time constructions of Ramanujan graphs [52]. However, the non-bipartite version of this question remained open. Our recent work with McKenzie and Yau [90] gives an affirmative answer: for sufficiently large N , approximately 69% of d -regular graphs on N vertices are Ramanujan, proved via edge universality for random d -regular graphs.

Quantum chaos. Bohigas–Giannoni–Schmit [33] conjectured that when the underlying classical dynamics are ergodic and sufficiently chaotic, the spectrum of the corresponding Schrödinger operator exhibits the same local statistics as those of a large random matrix from one of Wigner’s ensembles. In this setting, Berry’s random wave conjecture [29, 30] posits that, at small scales, high-energy eigenfunctions behave like a Gaussian random field. Geodesic flows on compact, boundaryless manifolds of negative sectional curvature provide rich families of chaotic systems. In this case the Schrödinger operator is the Laplace–Beltrami operator, and the *Quantum Unique Ergodicity* conjecture of Rudnick and Sarnak [126] predicts that the entire sequence of eigenfunctions becomes equidistributed as the eigenvalue $\lambda \rightarrow \infty$. These conjectures remain open in general; see [129, 15] for recent progress. By contrast, analogues of these statements for random d -regular graphs—often viewed as toy models of random hyperbolic surfaces—have been proved or partially proved. Such toy models are simpler due to their inherent discreteness and are also more amenable to numerical simulation. Understanding the scope and universality of the conjectures above is one of several motivations for studying random d -regular graphs.

Random surfaces. The first nonzero Laplace eigenvalue λ_1 (the spectral gap) of a compact hyperbolic surface quantifies “connectivity”: a large λ_1 means the surface is hard to cut, has small diameter, and mixes rapidly. For large surfaces, λ_1 is asymptotically bounded above by $1/4$ [94]. In analogy with Alon’s conjecture for random d -regular graphs, one can ask: do large random hyperbolic surfaces have an almost optimal spectral gap? Are there infinite sequences of hyperbolic surfaces with spectral gap larger than $1/4$? Recent breakthroughs, inspired by techniques for extreme eigenvalues of random d -regular graphs and random matrices, show that large random hyperbolic surfaces have, with high probability, an almost optimal spectral gap: $\lambda_1 \geq 1/4 - \varepsilon$ in the random lift model [85, 113, 83] and in the Weil–Petersson model [17, 18, 84].

1.4 Notation. For two quantities X_N and Y_N depending on N , we write that $X_N = O(Y_N)$ or $X_N \lesssim Y_N$ if there exists some universal constant such that $|X_N| \leq \mathfrak{C}Y_N$. We write $X_N = o(Y_N)$, or $X_N \ll Y_N$ if the ratio $|X_N|/Y_N \rightarrow 0$ as N goes to infinity. We write $X_N \asymp Y_N$ if there exists a universal constant $\mathfrak{C} > 0$ such that $Y_N/\mathfrak{C} \leq |X_N| \leq \mathfrak{C}Y_N$. For an $N \times N$ real symmetric matrix M , we denote its eigenvalues as $\lambda_1(M) \geq \lambda_2(M) \geq \dots \geq \lambda_N(M)$.

2 Background to Random Matrices. We survey results on random graphs from the point of view of random matrix theory. In this section we briefly discuss some background on random matrices. *Wigner*

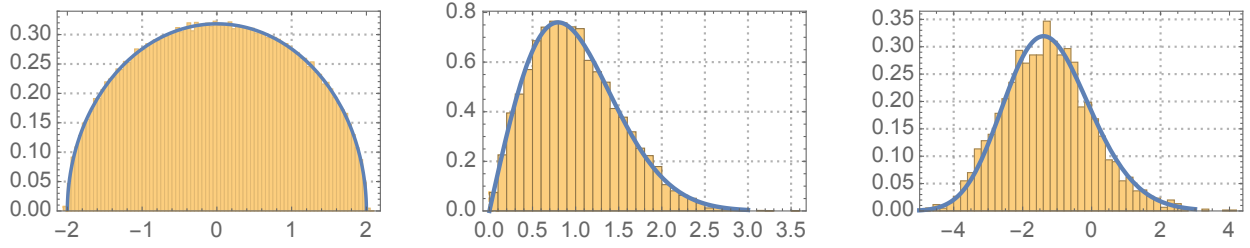


Figure 2.1: Shown to the left is the empirical eigenvalue distribution of a 500×500 GOE and the semicircle distribution; Shown to the middle is the limiting distribution of bulk eigenvalue gaps of GOE; Shown to the right is the Tracy-Widom distribution with $\beta = 1$, which is the limiting distribution of the rescaled largest eigenvalue of GOE.

matrices are introduced by Wigner, which are fundamental objects in the field. For conciseness, we consider (generalized) Wigner matrices in the orthogonal symmetry class. These are real symmetric $N \times N$ random matrices $H = (H_{ij})_{i,j=1}^N$ with $\{H_{ij} : 1 \leq i \leq j \leq N\}$ independent and centered. Let $\sigma_{ij}^2 := \text{Var}(H_{ij})$. There exists a constant $C > 0$ such that, $1/C \leq N \sigma_{ij}^2 \leq C$ for $1 \leq i, j \leq N$, and $\sum_{j=1}^N \sigma_{ij}^2 = 1$ for $1 \leq i \leq N$. We also assume subgaussian tails for $\sqrt{N} H_{ij}$ uniformly in i, j ; in most results below this can be relaxed to a suitable finite high-moment assumption. A canonical example is the Gaussian Orthogonal Ensemble (GOE).

As the size N of the matrix goes to infinity, the empirical eigenvalue distribution of a Wigner matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ converges almost surely to the semicircle distribution

$$(2.1) \quad \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i} \Rightarrow \varrho_{\text{sc}}(x) dx, \quad \varrho_{\text{sc}}(x) = \frac{\sqrt{4-x^2}}{2\pi} \mathbf{1}_{[-2,2]}(x).$$

The universality conjecture asserts that the limiting distributions of local spectral statistics for random matrices depend only on the symmetry class of the ensemble and not on the detailed distribution of its entries. Informally, after appropriate rescaling, the statistics behave as if the entries were Gaussian. We focus on three manifestations of universality.

Bulk universality. Fix an energy $E \in (-2, 2)$ in the interior of the spectrum. One studies local bulk observables, such as the k -point correlation functions of the rescaled eigenvalue process, or the rescaled gaps between consecutive bulk eigenvalues around E . The bulk universality conjecture states that these limiting local statistics depend only on the symmetry class, i.e. real symmetric or complex Hermitian, not on the entry distribution. For Wigner matrices this has been proved in [2, 65, 3, 41, 61, 67, 62, 66, 63, 135].

Edge universality. The largest eigenvalue fluctuates around 2 on the scale $N^{-2/3}$. After rescaling,

$$N^{2/3}(\lambda_1 - 2) \Rightarrow \text{TW}_1,$$

where TW_1 is the Tracy-Widom₁ distribution for the orthogonal symmetry class [139, 140]. Edge universality for Wigner matrices was established in [133, 67, 133, 134, 100, 109].

Eigenvector universality. For GOE matrices, orthogonal invariance implies that eigenvectors are Haar-distributed on the orthogonal group; in particular, each coordinate of a normalized eigenvector is asymptotically Gaussian. For generalized Wigner matrices, eigenvectors are also asymptotically Gaussian; see [43, 100, 136, 114].

3 Spectral Statistics of Erdős-Rényi Graphs. The global statistics of the eigenvalues of the Erdős-Rényi graph model $\mathbb{G}(N, p_N)$ are well understood. The empirical eigenvalue distribution of the normalized adjacency matrix H converges to the semi-circle distribution provided the average degree grows with the size of the graph, i.e. $N p_N \gg 1$, which follows from Wigner's original proof [142, 143]. Under the same regime, it was proven in [132] that the linear statistics, after proper normalization, converge to a Gaussian random variable.

For the largest eigenvalue, it has been proven in [101], for $\mathbb{G}(N, p_N)$ with adjacency matrix A , with probability tending to 1, $\lambda_1(A) = (1 + o(1)) \max\{\sqrt{\Delta}, N p_N\}$, where Δ is the maximum degree, and the $o(1) \rightarrow 0$ as

$\max\{\sqrt{\Delta}, Np_N\} \rightarrow \infty$. In the regime $Np_N \geq (\log N)^C$ for some fixed $C > 0$, the top eigenvalue separates from the rest of the spectrum and exhibits asymptotically Gaussian fluctuations [74, 60].

3.1 Bulk statistics. In this section we first discuss eigenvalue and eigenvector universality for graphs with polynomially growing degree; that is, for Erdős–Rényi graphs $\mathbb{G}(N, p_N)$ with $Np_N \geq N^c$ for any (arbitrarily small) exponent $c > 0$.

THEOREM 3.1 (Bulk eigenvalue universality [60, 59, 88, 104]). *Let H be the normalized adjacency matrices of Erdős–Rényi random graphs $\mathbb{G}(N, p_N)$ with polynomially growing degree, i.e. $Np_N \geq N^c$ for arbitrarily small $c > 0$. We recall the semi-circle distribution from (2.1), and let γ_i be the classical eigenvalue location defined by*

$$\frac{i}{N} = \int_{-2}^{\gamma_i} \varrho_{\text{sc}}(x) dx.$$

Fix $k \geq 1$ and any smooth, compactly supported $F : \mathbb{R}^k \rightarrow \mathbb{R}$, there exists a small $\varepsilon > 0$, the following two forms of bulk universality hold, for N sufficiently large,

1. *Gap universality in the bulk: For any $\kappa > 0$ and index i with $\kappa N \leq i \leq (1 - \kappa)N$,*

$$(3.1) \quad \begin{aligned} & \mathbb{E}_H [F(N\varrho_{\text{sc}}(\gamma_i)(\lambda_i - \lambda_{i+1}), \dots, N\varrho_{\text{sc}}(\gamma_i)(\lambda_i - \lambda_{i+k}))] \\ &= \mathbb{E}_{\text{GOE}} [F(N\varrho_{\text{sc}}(\gamma_i)(\mu_i - \mu_{i+1}), \dots, N\varrho_{\text{sc}}(\gamma_i)(\mu_i - \mu_{i+k}))] + O(N^{-\varepsilon}), \end{aligned}$$

where the second expectation is with respect to a GOE matrix with eigenvalues $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$.

2. *Fixed energy universality in the bulk. Denote the k -point correlation functions of H and of GOE by $\varrho_H^{(k)}$ and $\varrho_{\text{GOE}}^{(k)}$, respectively. Then for any $E \in (-2, 2)$,*

$$(3.2) \quad \begin{aligned} & \int_{\mathbb{R}^k} F(\alpha_1, \dots, \alpha_k) \frac{1}{\varrho_{\text{sc}}^k(E)} \varrho_H^{(k)} \left(E + \frac{\alpha_1}{N\varrho_{\text{sc}}(E)}, \dots, E + \frac{\alpha_k}{N\varrho_{\text{sc}}(E)} \right) d\alpha_1 \dots d\alpha_k \\ &= \int_{\mathbb{R}^k} F(\alpha_1, \dots, \alpha_k) \frac{1}{\varrho_{\text{sc}}^k(E)} \varrho_{\text{GOE}}^{(k)} \left(E + \frac{\alpha_1}{N\varrho_{\text{sc}}(E)}, \dots, E + \frac{\alpha_k}{N\varrho_{\text{sc}}(E)} \right) d\alpha_1 \dots d\alpha_k + O(N^{-\varepsilon}) \end{aligned}$$

The gap universality and the *averaged-energy* universality (i.e., (3.2) after averaging E over an interval of length $N^{-1+o(1)}$) were first proved for Erdős–Rényi graphs in the regime $Np_N \geq N^{2/3+c}$ [60, 59]. These results were later extended to the regime $Np_N \geq N^c$ in [88]. Moreover, Combining the approaches developed in these papers with the fixed-energy universality for Dyson Brownian motion established in [104] yields *fixed-energy* bulk universality as stated in (3.2). The proofs of all these results follow the three-step strategy, developed by Erdős–Yau and collaborators, for establishing bulk universality of Wigner matrices [64].

The following result states that bulk eigenvectors of sparse random graphs are asymptotically jointly Gaussian, provided the average degree grows polynomially in N .

THEOREM 3.2 (Bulk eigenvector universality [42, 114]). *Adopt the same setting as in Theorem 3.1. We denote the normalized eigenvectors of H as $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N$ (each multiplied by a random sign). Fix $k \geq 1$ and a polynomial F in k variables. For any unit vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_k \in \mathbb{R}^N$ orthogonal to $\mathbf{e} = (1, \dots, 1)^\top / \sqrt{N}$, and any deterministic indices $i_1, i_2, \dots, i_k$ with $\kappa N \leq i_a \leq (1 - \kappa)N$ for $a = 1, 2, \dots, k$, there exists a small $\varepsilon > 0$ the following statement holds for N sufficiently large,*

$$(3.3) \quad \mathbb{E}_H \left[P \left(\sqrt{N} \langle \mathbf{q}_1, \mathbf{u}_{i_1} \rangle, \dots, \sqrt{N} \langle \mathbf{q}_k, \mathbf{u}_{i_k} \rangle \right) \right] = \mathbb{E} [P(\mathcal{N}_1, \dots, \mathcal{N}_k)] + O(N^{-\varepsilon}),$$

where $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n$ are independent standard Gaussian random variables.

The special case of the above result when $\mathbf{q}_1 = \mathbf{q}_2 = \dots = \mathbf{q}_k$ has been proven in [42, Theorem 1.1]. The general setting as stated in Theorem 3.2 has been proven in [114, Theorem 2.9]. Both proofs involve analyzing the *eigenvector flow*—introduced in [43], which describes the Lie group valued flow of eigenvectors induced by matrix valued Brownian motion.

In the constant-average-degree regime, i.e., $Np_N \rightarrow \lambda \in (0, \infty)$, the semicircle law gives way to a non-explicit limiting measure μ_λ , first predicted in [121, 75] and later identified in [37] as the expected spectral measure of the

adjacency operator on a Poisson–Galton–Watson tree with mean offspring λ . We refer to [21, 36, 38, 127, 98, 58, 54] for analytic properties of μ_λ .

In this sparse regime, it was conjectured in [68] that there exists a critical value $\lambda_c > 1$ such that, for $Np_N > \lambda_c$, bulk eigenvalues are strongly correlated and exhibit GOE random-matrix statistics, whereas for $Np_N < \lambda_c$ they are asymptotically uncorrelated and follow Poisson statistics. This conjecture is supported by extensive numerical simulations [68, 103] and by supersymmetric methods [75, 121].

3.2 Edge statistics with polynomial-degree. In this section we discuss eigenvalue and eigenvector universality for Erdős–Rényi graphs $\mathbb{G}(N, p_N)$ with $Np_N \geq N^\epsilon$ for any (arbitrarily small) exponent $\epsilon > 0$.

THEOREM 3.3 (Edge eigenvalue universality [60, 59, 108, 89, 81, 107, 91, 79]). *Let H be the normalized adjacency matrix of the Erdős–Rényi random graph $\mathbb{G}(N, p_N)$ with polynomially growing average degree, i.e. $Np_N \geq N^\epsilon$ for some (arbitrarily small) $\epsilon > 0$. Fix $k \geq 1$. There exists an explicit random shift $\mathcal{L} = \mathcal{L}(H)$ (depending on certain averaged statistics of H) such that, for any smooth, compactly supported $F : \mathbb{R}^k \rightarrow \mathbb{R}$, there exists $\varepsilon > 0$ for which the following edge universality holds for all sufficiently large N :*

$$(3.4) \quad \begin{aligned} & \mathbb{E}_H \left[F \left(N^{2/3}(\lambda_2 - \mathcal{L}), \dots, N^{2/3}(\lambda_{k+1} - \mathcal{L}) \right) \right] \\ &= \mathbb{E}_{\text{GOE}} \left[F \left(N^{2/3}(\mu_1 - 2), \dots, N^{2/3}(\mu_k - 2) \right) \right] + O(N^{-\varepsilon}). \end{aligned}$$

Here $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ are the eigenvalues of H , and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$ are the eigenvalues of a GOE matrix. Moreover, the shift $\mathcal{L}$ decomposes into its mean and fluctuation parts, $\mathcal{L} = \mathbb{E}[\mathcal{L}] + \Delta\mathcal{L}$, with the first few terms of $\mathbb{E}[\mathcal{L}]$ given by

$$(3.5) \quad \mathbb{E}[\mathcal{L}] = 2 + \frac{1}{Np_N} - \frac{5}{4(Np_N)^2} + \dots,$$

and the leading fluctuation described by

$$(3.6) \quad \Delta\mathcal{L} \approx \frac{1}{N} \sum_{i,j} (H_{ij}^2 - \mathbb{E}[H_{ij}^2]),$$

which is of order $1/(N\sqrt{p_N})$. Analogous results hold for the smallest eigenvalues.

A version of Theorem 3.3 also holds for the centered normalized adjacency matrix H° . In that case, H° no longer has a large outlier eigenvalue corresponding to the average degree, and the edge universality holds for the top k eigenvalues; i.e. the left-hand side of (3.4) should be replaced by

$$\mathbb{E}_{H^\circ} \left[F \left(N^{2/3}(\lambda_1 - \mathcal{L}), \dots, N^{2/3}(\lambda_k - \mathcal{L}) \right) \right],$$

where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ are the eigenvalues of H° .

Edge universality for sparse random matrices was first proved in the regime $Np_N \geq N^{2/3+\epsilon}$ in [60, 59] with $\mathcal{L} = 2$, i.e. without a shift of the edge location. It was later extended to $Np_N \geq N^{1/3+\epsilon}$ in [108], yielding $\mathcal{L} = 2 + 1/(Np_N)$, which displays a deterministic shift of the edge.

Subsequently, it was observed in [89, 81] that there is a transition at $Np_N \asymp N^{1/3}$. More precisely, the centered second-largest eigenvalue satisfies

$$\lambda_2 - \mathbb{E}[\lambda_2] \approx \mathcal{X} + N^{-2/3} \text{TW}_1,$$

where $\mathcal{X} = (1/N) \sum_{i,j} (H_{ij}^2 - \mathbb{E}[H_{ij}^2])$ is the leading fluctuation appearing in Theorem 3.3 (asymptotically Gaussian of size $1/(N\sqrt{p_N})$), and TW_1 denotes the Tracy–Widom₁ distribution. Thus there is a *crossover* in the extreme-eigenvalue behavior at $Np_N \asymp N^{1/3}$: when $Np_N \gg N^{1/3}$, the Tracy–Widom term dominates and λ_2 has Tracy–Widom fluctuations, while for $Np_N \ll N^{1/3}$, the Gaussian term $\mathcal{X}$ dominates. At the critical scale $Np_N \asymp N^{1/3}$, the fluctuations are a mixture of the Gaussian and Tracy–Widom contributions.

A natural question is whether one can identify higher-order fluctuation terms so that, after subtracting all explicit corrections, edge universality (in the Tracy–Widom sense) holds uniformly throughout the regime

$Np_N \geq N^c$. This was partially settled in [107], which proved that for $Np_N \geq N^c$ there exists a sequence of explicit random correction terms capturing subleading fluctuations of the extreme eigenvalues; after subtracting their sum $\mathcal{L}$, the extreme eigenvalues concentrate on the optimal scale $N^{-2/3+o(1)}$. Finally, it has been shown that, up to this explicit random shift, the fluctuations of the extreme eigenvalues are Tracy–Widom distributed as stated in Theorem 3.3: first for the centered normalized adjacency matrix [91], and subsequently for the normalized adjacency matrix [79].

When the average degree grows polynomially with the graph size N , the edge eigenvector—like those in the bulk—are asymptotically jointly Gaussian as stated in the following theorem. This result, proved in [79], does not rely on analyzing the eigenvector flow or comparing with the GOE; instead, its key ingredient is an algorithm that directly computes the Fourier transform of the eigenvector distribution.

THEOREM 3.4 (Edge eigenvector universality [79]). *Adopt the same setting as in Theorem 3.3. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ denote the eigenvalues of H , and let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N$ be the associated normalized eigenvectors (each multiplied by a random sign). Fix $k \geq 1$ and any smooth, compactly supported $F : \mathbb{R}^k \mapsto \mathbb{R}$. For any unit vectors $\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_k \in \mathbb{R}^N$ orthogonal to $\mathbf{e} = (1, \dots, 1)^\top / \sqrt{N}$, there exists a small $\varepsilon > 0$ the following statement holds for N sufficiently large,*

$$(3.7) \quad \mathbb{E}_H \left[F \left(\sqrt{N} \langle \mathbf{q}_1, \mathbf{u}_2 \rangle, \dots, N \langle \mathbf{q}_k, \mathbf{u}_{k+1} \rangle \right) \right] = \mathbb{E} [F(\mathcal{N}_1, \dots, \mathcal{N}_k)] + O(N^{-\varepsilon}),$$

where $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_n$ are independent standard Gaussian random variables.

3.3 Edge statistics with logarithmic-degree. In this section, we discuss results on the edge statistics of Erdős–Rényi graphs $\mathbb{G}(N, p_N)$ in the logarithmic-degree regime. When $Np_N \gg \log N$, the spectral norm of the (centered and normalized) adjacency matrix of Erdős–Rényi graphs converges to 2; several results extend to inhomogeneous Erdős–Rényi models as well [28, 27, 106]. In this regime, all eigenvectors are delocalized [82, 60].

A sharp transition was later identified in [8, 138]: there exists a critical constant $b_* = 1/(\ln(4) - 1)$, such that if $\liminf_{N \rightarrow \infty} Np_N / \log N \geq b_*$, then all nontrivial (i.e., excluding λ_1) eigenvalues of the normalized adjacency matrix H satisfy $|\lambda_i(H)| \leq 2 + o(1)$; whereas if $\limsup_{N \rightarrow \infty} Np_N / \log N < b_*$, nontrivial edge eigenvalues (outside $[-2, 2]$) arise from high-degree “hub” vertices.

For $(\ln \ln N)^4 \ll Np \leq (b_* - (\log N)^{-\xi}) \ln N / N$ (any fixed $\xi > 0$), it was shown in [10] that the eigenvalues near the spectral edge form an asymptotic Poisson point process and the associated eigenvectors are exponentially localized. These results were extended in [86] to the range $Np_N \geq (\log N)^{-1/9}$, thereby covering constant-degree Erdős–Rényi graphs. For $\sqrt{\log N} \ll d_N \lesssim N$, a more complete picture of eigenvectors is established in [10, 7, 11, 9]: the spectrum exhibits three regimes—a delocalized bulk (eigenvectors completely delocalized), a fully localized edge (exponentially localized eigenvectors), and an intermediate semilocalized phase in between.

4 Spectral Statistics of Random d -Regular Graphs. By local weak convergence, the empirical eigenvalue density of random d -regular graphs converges to that of the infinite d -regular tree, which is known as the *Kesten–McKay distribution*; see [99, 118]. This density is given by

$$(4.1) \quad \varrho_d(x) := \mathbf{1}_{[-2, 2]} \left(1 + \frac{1}{d-1} - \frac{x^2}{d} \right)^{-1} \frac{\sqrt{4-x^2}}{2\pi}.$$

4.1 Bulk statistics. We start with bulk eigenvalue universality for random d -regular graphs with polynomially growing degree.

THEOREM 4.1 (Bulk eigenvalue universality [26, 23, 104]). *Let H be the normalized adjacency matrices of random d -regular graphs, with polynomially growing degree, i.e. $N^c \leq d \leq N^{2/3-c}$ for arbitrarily small $c > 0$. Then the statements (3.1) and (3.2) holds.*

The gap universality and the *averaged-energy* universality (i.e., (3.2) after averaging E over an interval of length $N^{-1+o(1)}$) were proved for random d -regular graphs with $N^c \leq d \leq N^{2/3-c}$ [26, 23]. Later it was extended to *fixed-energy* bulk universality as stated in (3.2) in [104].

Based on extensive numerical simulations, it was conjectured in [95] bulk universality holds for random d -regular graphs for fixed degree.

Conjecture. For any fixed degree $d \geq 3$, prove the bulk eigenvalues of random d -regular graphs on a large number of vertices have fluctuations which tend to those of GOE.

4.2 Edge statistics. The seminal works of Alon–Milman [6] and Alon [4] established a connection between the magnitude of the second-largest eigenvalue of a regular graph and its expansion properties. Alon [4] conjectured that, with high probability for a uniformly random d -regular graph, $\lambda_2(A) = 2\sqrt{d-1} + o(1)$, as $N \rightarrow \infty$. Alon’s conjecture was first proved by Friedman [71] (see [44, 72, 69] for earlier results). An alternative proof was later given in [34]; see also [125] for a different approach yielding a weaker bound.

A natural extension of Alon’s question to the regime where the degree d grows with the graph size N (and tends to infinity) was studied in [55, 45, 137, 53, 102]. These works proved that for $1 \ll d \leq N/2$, the nontrivial extremal eigenvalues are bounded by $O(\sqrt{d})$ for random d -regular graphs in both the uniform and permutation models. In this setting, [141, Conjecture 5.3] proposed the sharper asymptotic $\lambda_2(A) = (2 + o(1))\sqrt{d(1-d/N)}$, for degrees $d \leq N/2$ that grow with N . This conjecture was subsequently proved in [24] for $N^\epsilon \leq d \leq N^{2/3-\epsilon}$ and in [128] for all $(\log N)^{10} \ll d \leq N/2$.

In the rest of this section, we discuss fluctuations of the extreme eigenvalues of random d -regular graphs. We begin with the case where d grows with N . By symmetry, we may restrict attention to $d \leq N/2$; otherwise one can pass to the complement graph, which is an $(N-d-1)$ -regular graph, and translate all statements accordingly.

THEOREM 4.2 (Edge eigenvalue universality [24, 93, 78]). *Fix arbitrarily small $\epsilon > 0$, take $N^\epsilon \leq d \leq N^{1/3-\epsilon}$ or $N^{2/3+\epsilon} \leq d \leq N/2$, and define*

$$(4.2) \quad \mathcal{L} = 2 + \sqrt{\frac{d}{(N-d)N}}.$$

Then for any integer $k \geq 1$, and $s_1, s_2, \dots, s_k \in \mathbb{R}$, there exists a small constant $\epsilon > 0$ the following statement holds for N sufficiently large,

$$(4.3) \quad \begin{aligned} & \mathbb{P}_H \left(N^{2/3} \left(\frac{\lambda_2}{\sqrt{1-d/N}} - \mathcal{L} \right) \geq s_1, \dots, N^{2/3} \left(\frac{\lambda_{k+1}}{\sqrt{1-d/N}} - \mathcal{L} \right) \geq s_k \right) \\ &= \mathbb{P}_{\text{GOE}} \left(N^{2/3}(\mu_1 - 2) \geq s_1, \dots, N^{2/3}(\mu_k - 2) \geq s_k \right) + O(N^{-\epsilon}), \end{aligned}$$

Here $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ are the eigenvalues of H , and $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$ are the eigenvalues of a GOE matrix. Analogous results hold for the smallest eigenvalues.

The edge universality in Theorem 4.2 for $N^\epsilon \leq d \leq N^{1/3-\epsilon}$ has been proven in [24] for $N^{2/9+\epsilon} \leq d \leq N^{1/3-\epsilon}$, and generalized to the regime $N^\epsilon \leq d \leq N^{1/3-\epsilon}$ in [93]. In these cases, $d/N \leq N^{-2/3-\epsilon}$ is much smaller than the size of the Tracy-Widom fluctuation, one can take $\mathcal{L} = 2$, and simplify the left-hand side of (4.3) as

$$(4.4) \quad \mathbb{P}_H \left(N^{2/3}(\lambda_2 - 2) \geq s_1, \dots, N^{2/3}(\lambda_{k+1} - 2) \geq s_k \right).$$

The case that $N^{2/3+\epsilon} \leq d \leq N/2$ has been proven in [78], where the correction in $\mathcal{L}$ is not negligible. Compare with the edge universality result of Erdős–Rényi random graphs Theorem 3.3, random d -regular graphs are more rigid, the edge location $\mathcal{L}$ does not have a random shift.

Due to technical constraints, edge universality presently covers only the regimes $N^{2/9+\epsilon} \leq d \leq N^{1/3-\epsilon}$, and $N^\epsilon \leq d \leq N^{1/3-\epsilon}$. We believe that the statement in Theorem 4.2 holds as long as the degree d grows with the graph size N , which can grow arbitrarily slow, e.g. $d = \lfloor \log \log N \rfloor$.

For random d -regular graphs, the fixed-degree regime is more challenging than the growing-degree regime. Given that random regular graphs are typically almost Ramanujan, it has long been suggested that analyzing their typical behavior could produce examples of Ramanujan graphs [111, Problem 10.7.4]. Early simulations suggested that, the distribution of the second largest eigenvalue of random d -regular graphs after normalizing is the same as that of the largest eigenvalue of a matrix sampled from the GOE. Sarnak [130] and Miller–Novikoff–Sabelli [120] conjectured that, for every $d \geq 3$, there exist constants C_1^d and C_2^d such that

$$C_1^d N^{2/3}(\lambda_2 - 2) - C_2^d \Rightarrow \text{TW}_1.$$

An analogous statement holds for λ_N . In particular, this would imply that the extreme-eigenvalue fluctuations are of order $N^{-2/3+o(1)}$, and that a positive proportion of d -regular graphs are Ramanujan.

The proofs of Alon’s conjecture in [71, 34] proceed via high-moment bounds obtained by classifying closed walks by length and cycle structure; this yields the quantitative estimate $\lambda_2 \leq 2\sqrt{d-1} + O((\log \log N / \log N)^2)$. Subsequent work [25, 92] provided a new proof with a polynomially small error term, $\lambda_2 = 2\sqrt{d-1} + O(N^{-\varepsilon})$, for some small $\varepsilon > 0$, based on a careful analysis of the Green’s function and, in fact, establishing concentration for all eigenvalues. More recently, a new approach to strong convergence of random matrices [49, 50] gave, in the d -regular setting, the quantitative estimates $\lambda_2 \leq 2\sqrt{d-1} + O((\log N / N)^{1/6})$. Finally, our recent work with McKenzie and Yau [90] establishes that edge eigenvalues concentrates on the scale $N^{-2/3+o(1)}$ and edge universality holds for random d -regular graphs for every $d \geq 3$.

THEOREM 4.3 (Optimal eigenvalue rigidity [90]). *Fix $d \geq 3$, and an arbitrarily small $\mathfrak{c} > 0$. Define the classical location γ_i of the i -th eigenvalue of the Kesten-McKay distribution ϱ_d from (4.1) by*

$$\int_{\gamma_i}^2 \varrho_d(x) dx = \frac{i}{N}, \quad 2 \leq i \leq N.$$

There exists a positive integer $\omega_d \geq 1$, such that with probability $1 - N^{-(1-\mathfrak{c})\omega_d}$, the eigenvalues $\lambda_1 = d/\sqrt{d-1} \geq \lambda_2 \geq \dots \geq \lambda_N$ of the normalized adjacency matrix H of random d -regular graphs satisfy:

$$(4.5) \quad |\lambda_i - \gamma_i| \leq N^{-2/3+\mathfrak{c}} (\min\{i, N-i+1\})^{-1/3}, \text{ for every } 2 \leq i \leq N.$$

The above theorem establishes that edge eigenvalues concentrates on the scale $N^{-2/3+o(1)}$, and bulk eigenvalues (e.g. $i = N/2$) concentrates on the scale $N^{-1+o(1)}$, which are optimal and match those for Wigner matrices.

THEOREM 4.4 (Edge eigenvalue universality [90]). *Fix $d \geq 3$, $k \geq 1$ and $s_1, s_2, \dots, s_k \in \mathbb{R}$, and let $\mathcal{A} = d(d-1)/(d-2)^2$. There exists a small $\varepsilon > 0$ such that the eigenvalues $\lambda_1 = d/\sqrt{d-1} \geq \lambda_2 \geq \dots \geq \lambda_N$ of the normalized adjacency matrix H of random d -regular graphs satisfy:*

$$\begin{aligned} & \mathbb{P}_H \left((AN)^{2/3}(\lambda_2 - 2) \geq s_1, \dots, (AN)^{2/3}(\lambda_{k+1} - 2) \geq s_k \right) \\ &= \mathbb{P}_{\text{GOE}} \left(N^{2/3}(\mu_1 - 2) \geq s_1, \dots, N^{2/3}(\mu_k - 2) \geq s_k \right) + O(N^{-\varepsilon}), \end{aligned}$$

where $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$ are the eigenvalues of the GOE. The analogous statement holds for the smallest eigenvalues $-\lambda_N, \dots, -\lambda_{N-k+1}$.

Theorem 4.4 implies that the fluctuation of the second largest eigenvalue converges to the Tracy-Widom₁ distribution. The Tracy-Widom₁ distribution has about 83% of its mass on the set $\{x : x < 0\}$ [48]. Therefore Theorem 4.4 implies 83% of d -regular graphs have the second eigenvalue *less than* 2. The proof of Theorem 4.4 can be extended to show that the largest and smallest nontrivial eigenvalues converge in distribution to *independent* Tracy-Widom₁ distributions. As a consequence, we have the following result.

COROLLARY 4.5 ([90]). *Fix $d \geq 3$ and N sufficiently large. With probability approximately 69%, a randomly sampled d -regular graph has $\max\{\lambda_2, |\lambda_N|\} \leq 2$, and is therefore Ramanujan.*

Random d -regular graphs can also be constructed from random lifts of a base graph containing two vertices and d edges between them. More generally, one can study the random N -lift $\tilde{\mathcal{G}}$ of an arbitrary base graph $\mathcal{G}$. All such lifts share the same universal cover, namely the universal cover of $\mathcal{G}$. The lifted graph $\tilde{\mathcal{G}}$ inherits every eigenvalue of $\mathcal{G}$ (the *old* eigenvalues); all other eigenvalues are the *new* eigenvalues.

This class of random graphs obtained from random lifts—and in particular their new eigenvalues—has been extensively studied [12, 13, 70, 32, 73, 125, 110, 35, 49, 50]. In analogy with Alon’s conjecture, [70] conjectured that the new eigenvalues of random N -lifts are bounded by $\rho + o(1)$, where ρ is the spectral radius of the universal cover of the base graph; this was proved in [35, 49, 50].

A finite graph (not necessarily regular) is called *Ramanujan* if, aside from the “trivial” eigenvalues λ_1 and—when the graph is bipartite— $\lambda_N = -\lambda_1$, every eigenvalue lies in the interval $[-\rho, \rho]$. It is natural to ask whether *edge universality* holds for the new eigenvalues of random N -lifts of an arbitrary base graph $\mathcal{G}$. A positive answer would imply that, for large N , many such lifts are (irregular) Ramanujan graphs.

4.3 Eigenvectors of d -regular graphs. Random d -regular graphs share fundamental characteristics with classically *chaotic systems*, including strong expansion properties and delocalized eigenvectors. We begin by surveying results on eigenvectors of d -regular graphs, both random and deterministic, that point toward delocalization.

For (deterministic) locally tree-like regular graphs on vertex set $\llbracket N \rrbracket = \{1, 2, \dots, N\}$, it has been shown that eigenvectors $\mathbf{u}$ are *weakly delocalized* in the sense that $\|\mathbf{u}\|_\infty \leq (\log N)^{-\mathfrak{c}} \|\mathbf{u}\|_2$, for some small $\mathfrak{c} > 0$, see [56, 76]. Moreover, eigenvectors cannot concentrate on small sets: if a vertex set $\mathbb{V} \subset \llbracket N \rrbracket$ satisfies $\sum_{i \in \mathbb{V}} |\mathbf{u}(i)|^2 \geq \varepsilon \|\mathbf{u}\|_2^2$, then $\mathbb{V}$ must contain at least $N^{\mathfrak{c}(\varepsilon)}$ vertices [47], for some small $\mathfrak{c}(\varepsilon)$ depending on ε .

For a sequence of deterministic, locally tree-like, regular expander graphs on N vertices, a *quantum ergodicity* statement is known: for any $\mathbf{a} \in \mathbb{R}^N$ with $\|\mathbf{a}\|_\infty \leq 1$ and $\sum_{i=1}^N \mathbf{a}(i) = 0$, and for any (possibly shrinking) spectral window $I_N \subset \mathbb{R}$, the average over eigenvectors $\mathbf{u}_j$ with eigenvalues $\lambda_j \in I_N$ satisfies

$$(4.6) \quad \frac{1}{|\{j : \lambda_j \in I_N\}|} \sum_{\lambda_j \in I_N} \left| \sum_{i=1}^N \mathbf{a}(i) \mathbf{u}_j^2(i) \right|^2 = o(1)$$

as $N \rightarrow \infty$ [16, 46, 20, 19, 14]. In these works, more general statements have been proved, in which the test function $\mathbf{a}$ in (4.6) is replaced by suitable operators, and the sequence of graphs may be irregular, provided the local weak limit exists.

For *random* d -regular graphs on N vertices, complete delocalization has been proved: with high probability, every eigenvector $\mathbf{u}$ satisfies $\|\mathbf{u}\|_\infty \leq (\log N)^C N^{-1/2} \|\mathbf{u}\|_2$, for some large $C > 0$; see [92, 25]. In addition, a probabilistic form of *quantum unique ergodicity* holds for these graphs: the bound (4.6) holds *without* averaging over many eigenvectors (again, with high probability over the choice of the graph).

In the remainder of this section, we focus on the distribution of eigenvectors for random d -regular graphs. The following result states that, the same as for Erdős-Rényi graph, bulk eigenvectors of random d -regular graphs are asymptotically jointly Gaussian, if the average degree grows polynomially in N .

THEOREM 4.6 (Bulk eigenvector universality [42, 114]). *Fix arbitrarily small $\mathfrak{c} > 0$, take $N^{\mathfrak{c}} \leq d \leq N^{1/3-\mathfrak{c}}$. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$ denote the eigenvalues of H , and let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N$ be the associated normalized eigenvectors (each multiplied by a random sign). Then the statement (3.7) holds.*

The special case of the above result when $\mathbf{q}_1 = \mathbf{q}_2 = \dots = \mathbf{q}_k$ has been proven in [42, Theorem 1.1]. The general setting as stated in Theorem 4.6 has been proven in [114, Theorem 2.9].

For random d -regular graphs with fixed degree d , numerical experiments reported in [87, §7.3.4] indicate that eigenvectors are *not* asymptotically uniform on the unit sphere. That survey posed the problem of identifying the limiting distribution of eigenvectors associated with the second-largest eigenvalue (in absolute value). It is now understood that the correct limiting object is the *Gaussian wave* on the d -regular tree; we describe this model below. This can be viewed as a graph-theoretic analogue of Berry's Gaussian wave conjecture [29, 30].

For any $d \geq 3$, let $\mathcal{X}_d$ denote the infinite d -regular tree, and o as a distinguished vertex, referred to as the root. The vertex set of $\mathcal{X}_d$ is denoted by $\mathbb{X}_d$. An *eigenvector process* with eigenvalue λ is a joint distribution $\{\psi_\lambda(i)\}_{i \in \mathbb{X}_d}$ of real valued random variables, each with variance 1 such that the following eigenvector equation holds almost surely

$$\sum_{i \in \mathbb{X}_d : i \sim o} \psi_\lambda(i) = \lambda \psi_\lambda(o),$$

where the summation is over all neighbors i of o . Furthermore, the process is invariant under any automorphism of the d -regular tree.

A *Gaussian wave* is an eigenvector process whose joint distribution is Gaussian. It is established in [57] that for every $-d \leq \lambda \leq d$ there exists a unique Gaussian wave ψ_λ with eigenvalue λ . In [22], Backhausz and Szegeedy showed that the *almost eigenvectors* of random regular graphs converge to *Gaussian waves*, with variance $0 \leq \sigma^2 \leq 1$. Having $\sigma < 1$ is the sign of a very (partial) scarring property, and this is believed not to happen for eigenvectors. Our result [80] jointly with He and Yau, gives an alternative proof for the edge eigenvectors of random regular graphs, demonstrating that the variance attains the full value $\sigma^2 = 1$ —i.e., no scarring in this sense.

THEOREM 4.7 ([80]). *Fix $d \geq 3$, and set $\mathcal{A} = d(d-1)/(d-2)^2$. Fix $k \geq 1$ and radius $r \geq 0$. Let A be the adjacency matrix of random d -regular graphs $\mathcal{G}$ on N vertices. Let $\lambda_1(A) = d \geq \lambda_2(A) \geq \dots \geq \lambda_N(A)$ denote the eigenvalues, and let $\mathbf{u}_1 = \mathbf{1}/\sqrt{N}, \mathbf{u}_2, \dots, \mathbf{u}_N$ be the associated normalized eigenvectors. Fix any vertex $o \in \llbracket N \rrbracket$. The eigenvectors $\{\mathbf{u}_s\}_{2 \leq s \leq k+1}$ (each multiplied by a random sign) when restricted to the radius- r ball $\mathcal{B}_r(o; \mathcal{G})$ on the graph $\mathcal{G}$, jointly converge to independent Gaussian waves:*

$$(\sqrt{N}\mathbf{u}_2|_{\mathcal{B}_r(o; \mathcal{G})}, \sqrt{N}\mathbf{u}_3|_{\mathcal{B}_r(o; \mathcal{G})}, \dots, \sqrt{N}\mathbf{u}_{k+1}|_{\mathcal{B}_r(o; \mathcal{G})}) \Rightarrow (Z_1|_{\mathcal{B}_r(o; \mathcal{X}_d)}, Z_2|_{\mathcal{B}_r(o; \mathcal{X}_d)} \dots, Z_k|_{\mathcal{B}_r(o; \mathcal{X}_d)}),$$

where $Z_1, Z_2, \dots, Z_k$ are independent copies of the Gaussian wave $\psi_{2\sqrt{d-1}}$ corresponding to $\lambda = 2\sqrt{d-1}$. An analogous result holds for the eigenvectors associated with the smallest eigenvalues.

The statement in [80] is slightly stronger than that of Theorem 4.7. It establishes that the edge eigenvalues and their eigenvectors are asymptotically independent: the former converge to the Airy₁ point process (from GOE [139, 140]), while the latter converge to Gaussian waves.

5 Methodology. We now outline the ideas behind proving edge universality for sparse Erdős–Rényi graphs $\mathbb{G}(N, p_N)$ and random d -regular graphs, as developed in [90, 93, 91, 92, 107]. For the normalized adjacency matrix H of these sparse graphs, edge universality follows the standard three-step scheme [64]. The key deviations arise in steps (i) and (iii):

- (i) In the sparse setting, the optimal concentration of eigenvalue locations is not with respect to the semicircle law as in the Wigner case. The self-consistent equations for the relevant Stieltjes transforms are therefore more intricate. New techniques—local resampling and higher-order cumulant expansions that more effectively exploit the randomness—are introduced to derive these equations and establish concentration; see Subsection 5.1 for details.
- (iii) For random d -regular graphs with fixed d , standard Wigner-type comparison methods fail. A different mechanism based on loop equations plays a central role; see Subsection 5.2 for details.

5.1 Three-step strategy. In this section we recall the three-step strategy for proving edge universality for Wigner matrices (see, e.g., [64]) and highlight the key differences that arise in the sparse random graph setting. Let H denote either a Wigner matrix or the normalized adjacency matrix of an Erdős–Rényi graph or a random d -regular graph. We write its eigenvalues in decreasing order as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$.

Step (i): local law and rigidity. The first step is to prove optimal eigenvalue rigidity up to the spectral edge (on the $N^{-2/3}$ scale), yielding precise control of the extreme eigenvalues. This is achieved by deriving a self-consistent equation for the Stieltjes transform $s_N(z)$ (or related spectral observables),

$$(5.1) \quad s_N(z) := \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z}.$$

For Wigner matrices, the self-consistent equation takes the form

$$(5.2) \quad \mathbb{E}[1 + z s_N(z) + s_N^2(z)] \approx 0,$$

whose unique solution is the Stieltjes transform of the semicircle law. For Erdős–Rényi graphs $\mathbb{G}(N, p_N)$ with polynomially growing degree $Np_N \geq N^c$, the self-consistent equation becomes a higher-degree polynomial,

$$(5.3) \quad \mathbb{E}[1 + z s_N(z) + s_N^2(z) + (a_2 s_N^2(z) + a_4 s_N^4(z) + \dots + a_{2L} s_N^{2L}(z))] \approx 0,$$

where L is a sufficiently large integer, and the coefficients $a_{2\ell}$ for $1 \leq \ell \leq L$ are (random) explicit polynomials in the entries of H . Moreover, with high probability they are small: $|a_{2\ell}| \leq N^{o(1)}/\sqrt{Np_N}$. For random d -regular graphs with polynomially growing degree, an equation of the form (5.3) also holds, except that now the coefficients $a_{2\ell}$ are deterministic.

For a random d -regular graph $\mathcal{G}$ on N vertices with fixed degree d , obtaining a closed equation for the Stieltjes transform requires a more refined observable. Let $\mathcal{G}^{(i)}$ be the graph obtained by removing vertex i . Let $H^{(i)}$

denote its normalized adjacency matrix, and set $G^{(i)}(z) := (H^{(i)} - z)^{-1}$ for the corresponding Green's function. Consider the average over adjacent vertices $i \sim j$,

$$(5.4) \quad Q(z) := \frac{1}{Nd} \sum_{i \sim j} G_{jj}^{(i)}(z),$$

where $i \sim j$ denotes adjacency in $\mathcal{G}$. The quantity $G_{jj}^{(i)}(z)$ in (5.4) can be approximated by the Green's function at the root of a truncated $(d-1)$ -ary tree of depth $\ell \asymp \log_{d-1} N$ with boundary weights $Q(z)$; we denote this approximation by $Y_\ell(Q(z), z)$. This yields the following self-consistent equation for $Q(z)$:

$$(5.5) \quad \mathbb{E}[Q(z) - Y_\ell(Q(z), z)] \approx 0.$$

The Stieltjes transform $s_N(z)$ of the empirical eigenvalue distribution of H can be recovered from $Q(z)$ via

$$(5.6) \quad \mathbb{E}[s_N(z) - X_\ell(Q(z), z)] \approx 0,$$

where $X_\ell(Q(z), z)$ denotes the Green's function at the root of a truncated d -regular tree of depth ℓ with boundary weights $Q(z)$.

For Wigner matrices and adjacency matrices of Erdős–Rényi graphs (with independent entries), equations such as (5.2) and (5.3) follow from integration by parts and, more generally, cumulant expansions up to arbitrarily high order [79, 108, 91, 107].

For adjacency matrices of random d -regular graphs, the fixed-degree constraint induces long-range correlations among matrix entries. To handle this, we employ *simple switchings*: replace two disjoint edges $\{v_1, v_2\}$ and $\{v_3, v_4\}$ by $\{v_1, v_3\}$ and $\{v_2, v_4\}$ (whenever the latter pairs are non-edges); this operation preserves the uniform measure on d -regular graphs. In practice, simple switchings are used to resample the boundary edges of large graph neighborhoods, a procedure known as *local resampling* (introduced in [25, 92]), which preserves the local tree-like structure while retaining sufficient global information. The graphs before and after local resampling have the same law and form an exchangeable pair, which can be used to perform discrete integration by parts when computing expectations. By iterating local resamplings and carefully tracking the errors, one obtains the self-consistent relations (5.5) and (5.6).

Step (ii): strong ergodicity of Dyson Brownian motion. The second step studies the normalized adjacency matrix perturbed by a small GOE matrix

$$(5.7) \quad H(t) := H + \sqrt{t} Z,$$

where Z is a GOE matrix for Wigner matrices and Erdős–Rényi graphs; for random d -regular graphs, Z is the GOE restricted to the subspace of matrices with vanishing row and column sums (so that the trivial eigenvector $\mathbf{e} = (1, \dots, 1)^\top$ is preserved).

Assume the eigenvalues of H satisfy optimal edge rigidity with respect to a limiting density ϱ exhibiting square-root edge behavior, which is the consequence of Step (i). Then, by [105, 1], after a time $t \geq N^{-1/3+\mathfrak{c}}$ (for arbitrarily small $\mathfrak{c} > 0$) the edge statistics of $H(t)$ are universal: the extreme eigenvalues converge to the Airy₁ point process (the edge scaling limit of GOE). In particular, for (generalized) Wigner matrices ϱ is the semicircle law (2.1); for Erdős–Rényi graphs, ϱ is random and characterized by the self-consistent equation (5.3); and for random d -regular graphs ϱ is the Kesten–McKay law ϱ_d from (4.1). The asymptotic empirical eigenvalue distribution of the matrix $H(t) = H + \sqrt{t} Z$ can be described by the free additive convolution, from free probability theory [31].

Step (iii): Green function comparison. This final step removes the Gaussian component in (5.7) and, via a comparison argument, transfers edge universality from the Gaussian-divisible model to the original ensemble. In particular, it shows that the k -point correlation function at the edge is invariant along the flow (5.7), with t interpreted as time. This is achieved by the following comparison theorem for multipoint correlation functions at the edge (equivalently, by comparing products of Stieltjes transforms on the microscopic scale). For details on how this theorem yields edge universality, see [64, Section 17].

We denote by ϱ_t the limiting empirical spectral distribution of $H(t)$ and write $E_t := \sup \text{supp } \varrho_t$ for its right edge. For $z \in \mathbb{C} \setminus \mathbb{R}$, let

$$G(z; t) := (H(t) - z)^{-1}, \quad s_N(z; t) := \frac{1}{N} \text{Tr } G(z; t), \quad s(z; t) := \int_{\mathbb{R}} \frac{\varrho_t(x)}{x - z} dx,$$

be the normalized trace of the Green's function and its deterministic counterpart (the Stieltjes transform of ϱ_t), respectively.

PROPOSITION 5.1. *Fix a small $\mathfrak{c} > 0$. We introduce the microscopic window:*

$$(5.8) \quad \mathbf{M} := \{w \in \mathbb{C} : N^{-2/3-\mathfrak{c}} \leq |\operatorname{Im}[w]| \leq N^{-2/3+\mathfrak{c}}, -N^{-2/3+\mathfrak{c}} \leq \operatorname{Re}[w] \leq N^{-2/3+\mathfrak{c}}\}.$$

Then for any $p \geq 1$, and $w_1, w_2, \dots, w_p \in \mathbf{M}$, $z_j(t) = E_t + w_j$ for $1 \leq j \leq p$, the following holds

$$(5.9) \quad \mathbb{E} \left[\prod_{j=1}^p N^{1/3} (s_N(z_j(t); t) - s(z_j(t); t)) \right] \Bigg|_{t=0}^{t=N^{-1/3+\mathfrak{c}}} = O(N^{-\mathfrak{c}}),$$

provided N is sufficiently large.

For Wigner matrices and random graphs with polynomially growing degree, Proposition 5.1 follows by integrating out the GOE perturbation Z via Gaussian integration by parts, expanding the randomness of H via cumulant expansion, and showing that the leading contributions cancel. For random d -regular graphs with fixed degree d , this argument breaks down. In the proof of Theorem 4.4, a key observation is that the time derivative of the multi-point correlation functions of the Stieltjes transform of $H(t)$ is governed by microscopic loop equations. In the next section, we state these loop equations and explain how they yield the Green function comparison for fixed-degree d -regular graphs.

5.2 Loop equations and Green's function comparison. Loop (or Dyson-Schwinger) equations were first used in the theoretical physics literature (e.g., in the work of Migdal [119]) and were later introduced to the mathematical community by Johansson [96] to derive macroscopic central limit theorems for general β -ensembles of eigenvalues of random matrices, see also [39, 40, 131] and the book [77]. For the Gaussian β -ensemble, the loop (or Dyson-Schwinger) equation describes a recursive structure satisfied by the Stieltjes transform (correlation functions) of the eigenvalues. The first loop equation is given by : for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$(5.10) \quad \mathbb{E} \left[s_N(z)^2 + z s_N(z) + 1 + \frac{2 - \beta}{\beta} \frac{\partial_z s_N(z)}{N} \right] = 0, \quad s_N(z) := \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z}.$$

More generally, for any $p \geq 1$, the loop equations at rank p are given by : for $z, z_1, z_2, \dots, z_{p-1} \in \mathbb{C} \setminus \mathbb{R}$

$$(5.11) \quad \mathbb{E} \left[\left(s_N(z)^2 + z s_N(z) + 1 + \frac{2 - \beta}{\beta} \frac{\partial_z s_N(z)}{N} \right) \prod_{j=1}^{p-1} s_N(z_j) \right] + \frac{2}{\beta N^2} \sum_{j=1}^{p-1} \mathbb{E} \left[\partial_{z_j} \frac{s_N(z) - s_N(z_j)}{z - z_j} \prod_{i \neq j} s_N(z_i) \right] = 0.$$

These loop equations (5.10) and (5.11) arise from the invariance of the eigenvalue integral under infinitesimal reparametrizations of the variables; they can be derived either by integration by parts or, equivalently, by an infinitesimal change of variables. More generally, one can derive the loop equations for β -ensembles with general potential V .

In general, loop equations are model-dependent. For β -ensembles, they depend on the potential V . However, when zooming in near the spectral edge, the microscopic version of loop equations becomes universal. To introduce *microscopic version of loop equations*, we recall that the extreme eigenvalues of GOE converge to the Airy_1 point process $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \dots$. We can introduce the following normalized Stieltjes transform of the Airy_1 point process:

$$Y(w) = \sum_{i \geq 1} \frac{1}{\Lambda_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\text{Ai}'(0)}{\text{Ai}(0)}, \quad w \in \mathbb{C} \setminus \mathbb{R},$$

where $\mathfrak{a}_1 > \mathfrak{a}_2 > \mathfrak{a}_3 > \dots$ are zeros of the Airy function $\text{Ai}(w)$. Under the above normalization, $Y(w)$ has square root behavior: $Y(w) \sim \sqrt{w}$ as $w \rightarrow \infty$. $Y(w)$ can be viewed as a random meromorphic function, with poles given by the Airy_1 point process.

Formally, taking the soft-edge scaling limit of the loop equations (5.10) and (5.11), we obtain a hierarchy of equations for the Stieltjes transform $Y(w)$ of the Airy_1 point process. The rank- p loop equations are given by: for any $w, w_1, \dots, w_{p-1} \in \mathbb{C} \setminus \mathbb{R}$,

$$(5.12) \quad \mathbb{E} \left[\left((Y(w) - \sqrt{w})^2 + 2\sqrt{w}(Y(w) - \sqrt{w}) + \partial_w Y(w) \right) \prod_{i=1}^{p-1} Y(w_i) \right] \\ + 2 \sum_{1 \leq i \leq p-1} \mathbb{E} \left[\partial_{w_i} \left(\frac{Y(w) - Y(w_i)}{w - w_i} \right) \prod_{j:j \neq i} Y(w_j) \right] = 0.$$

These loop equations (5.12) provide necessary conditions for the edge statistics to converge to the Airy point process.

A key intermediate step in [90] toward the proof of Theorem 4.4 is the following microscopic loop equation for the normalized adjacency matrix of a random d -regular graph perturbed by a small GOE. Its proof occupies the main part of [90] and is analogous to the derivation of the self-consistent equations (5.5) and (5.6). To make the error terms sufficiently small, we need to iterate the local resampling procedure many times and carefully track the accumulated errors.

PROPOSITION 5.2. *Fix a small $\mathfrak{c} > 0$. Let $\Delta_t(z) := s_N(z; t) - s(z; t)$, and recall the microscopic window $\mathbf{M}$ from (5.8). Fix $p \geq 1$ and $w, w_1, \dots, w_{p-1} \in \mathbf{M}$, and set $z(t) := E_t + w$ and $z_j(t) := E_t + w_j$ for $1 \leq j \leq p-1$. Then for N sufficiently large,*

$$(5.13) \quad \mathbb{E} \left[\left(\Delta_t(z(t))^2 + 2\mathcal{A}\sqrt{z(t) - E_t} \Delta_t(z(t)) + \frac{\partial_z s_N(z(t); t)}{N} \right) \prod_{i=1}^{p-1} \Delta_t(z_i(t)) \right] \\ + \frac{2}{N^2} \sum_{1 \leq i \leq p-1} \mathbb{E} \left[\partial_{z_i(t)} \left(\frac{s_N(z(t); t) - s_N(z_i(t); t)}{z(t) - z_i(t)} \right) \prod_{j:1 \leq j \neq i \leq p-1} \Delta_t(z_j(t)) \right] = O \left(N^{-(p+1)/3-10\mathfrak{c}} \right).$$

The scaling limit $N^{1/3} \Delta_t(E_t + w/(\mathcal{A}N)^{2/3})/\mathcal{A}^{2/3}$ converges to $Y(w) - \sqrt{w}$ in the microscopic version of loop equations (5.12). The error bound on the right-hand side of (5.13) is precisely calibrated so that, after scaling each factor of Δ_t by $N^{1/3}$, the total error still vanishes as $N \rightarrow \infty$. The microscopic loop equations (5.13) imply the Green's function comparison in Proposition 5.1, thereby completing the final step of the three-step strategy.

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