

Notes on Edge Universality of Random Regular Graphs

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Abstract

The purpose of this note is to expose the structure, general strategy, and main ideas of the proof in [15]. The core of the argument is the derivation of the self-consistent equations and the microscopic version of the loop equations for random d -regular graphs. We first recall the local law for random d -regular graphs, and then illustrate the main ideas behind the derivation of the self-consistent equations and the first loop equation.

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1 Introduction

We give an exposition of the ideas and proofs in [15]. The overall strategy consists of three main components:

1. proving concentration for the self-consistent equations, via estimates of the expectations of high moments of these equations, leading to optimal concentration of eigenvalues;
2. proving a microscopic version of the loop equations, which can be viewed as a refinement of the self-consistent equations with an additional correction term; and
3. deriving edge universality using the loop equations together with Dyson Brownian motion.

We start in Section 2 by recalling some useful identities for the Green's function, which are standard tools in random matrix theory. In Section 3, we recall the local law for random d -regular graphs and related concepts from the previous work [17].

To illustrate the basic ideas, in this note we only prove estimates for the expectation (first moment) of the self-consistent equations and derive the first microscopic version of the loop equation. These are stated in Theorem 4.5. We then derive edge universality using the loop equations in Section 4. The proof of Theorem 4.5 occupies the remainder of the note.

The key tool is the local resampling procedure in Section 5, which efficiently exploits the randomness of random d -regular graphs. Local resampling produces an exchangeable pair of random d -regular graphs (before and after resampling). In Section 6, we explain the basic ideas of proving concentration using exchangeable pairs, and in Section 7 we discuss how to apply this method in the setting of random d -regular graphs. It turns out, however, that the resulting error is too weak and far from optimal.

To derive self-consistent equations with optimal error, we need to:

1. develop an iteration mechanism, in which we repeatedly perform local resampling and the error improves at each step (see Section 11);
2. track the terms arising from the iteration by introducing admissible functions and forests in Section 8;
3. for terms depending on the graph after local resampling, rewrite them as functions of the original graph using either the Schur complement formula (in Section 9) or the Woodbury formula (in Section 10); and
4. control the errors generated at each iteration step, as done in Section 13.

We remark that the derivation of the estimates for the self-consistent equations and the loop equation are not independent. The loop equation (as proved in Section 12) essentially follows from identifying the next order correction to the self-consistent equations.

1.1 Random d -regular graphs

A graph on N vertices $\llbracket N \rrbracket = \{1, \dots, N\}$ is d -regular if every vertex has degree d . Necessarily $0 \leq d \leq N-1$ and dN is even. In this article we mainly consider the *uniform model* $\mathcal{G}_{N,d}$: Sample uniformly from the finite set of all *simple* (no loops or multiple edges) d -regular graphs on N vertices $\llbracket N \rrbracket$.

Besides the uniform model, several other models are considered in the literature.

Permutation model. When d is even, take $d/2$ independent uniformly distributed permutations $\sigma_1, \dots, \sigma_{d/2}$ on $\llbracket N \rrbracket$. The permutation model is the random graph on N vertices obtained by adding an edge $\{i, \sigma_j(i)\}$ for each $i \in \llbracket N \rrbracket$ and $1 \leq j \leq d/2$.

Matching model. When N is even, take d independent uniformly distributed perfect matching $\sigma_1, \dots, \sigma_d$ on $\llbracket N \rrbracket$. The matching model is the random graph on N vertices obtained by adding an edge $\{i, \sigma_j(i)\}$ for each $i \in \llbracket N \rrbracket$ and $1 \leq j \leq d$.

These models yield a d -regular *multigraph* with possible self-loops and multiple edges; conditioning on simplicity gives the uniform model $\mathcal{G}_{N,d}$. In this article we mainly discuss the uniform model, but many results also hold for other models.

We can identify a d -regular graph with its adjacency matrix A . Let $\mathcal{G}$ be a simple, undirected random graph on the vertex set $\llbracket N \rrbracket := \{1, \dots, N\}$. Its adjacency matrix $A = (A_{ij}) \in \{0, 1\}^{N \times N}$ is defined by $A_{ij} = 1$ if and only if i and j are adjacent. For d -regular graphs, it is immediate that A has a trivial eigenvalue d with associated eigenvector $\mathbf{e} = (1, 1, \dots, 1)^\top$. Moreover, by the Perron-Frobenius theorem, all other eigenvalues are bounded in absolute value by d . For convenience, we shall consider the normalized adjacency matrix

$$H := A/\sqrt{d-1}. \quad (1.1) \quad \{\text{def}_H\}$$

We denote the eigenvalues of the normalized adjacency matrix H of a d -regular graph $\mathcal{G}$ on N vertices as $\lambda_1 = d/\sqrt{d-1} \geq \lambda_2 \geq \dots \geq \lambda_N$. We also introduce the Green's function and the Stieltjes transform of the empirical eigenvalue distribution of the normalized adjacency matrix H

$$G(z) := (H - z)^{-1}, \quad m_N(z) := \frac{1}{N} \sum_i G_{ii}(z) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z}, \quad z \in \mathbb{C}^+. \quad (1.2) \quad \{\text{e:G}\}$$

By local weak convergence, the empirical eigenvalue density of random d -regular graphs converges to that of the infinite d -regular tree, which is known as the Kesten-McKay distribution; see [19, 21]. This density is given by

$$\varrho_d(x) := \mathbf{1}_{x \in [-2, 2]} \left(1 + \frac{1}{d-1} - \frac{x^2}{d} \right)^{-1} \frac{\sqrt{4-x^2}}{2\pi}. \quad (1.3) \quad \{\text{e:KMdistrib}\}$$

Note that close to the spectral edge ± 2 , the Kesten-McKay distribution has square root behavior:

$$x \rightarrow \pm 2, \quad \varrho_d(x) = \frac{\mathcal{A}\sqrt{2 \mp x}}{\pi} + O(|2 \mp x|), \quad \mathcal{A} := \frac{d(d-1)}{(d-2)^2}. \quad (1.4) \quad \{\text{e:edge_behav}\}$$

We denote by $m_d(z)$ the Stieltjes transform of the Kesten-McKay distribution $\varrho_d(x)$,

$$m_d(z) = \int_{\mathbb{R}} \frac{\varrho_d(x) dx}{x - z}, \quad z \in \mathbb{C}^+ := \{w \in \mathbb{C} : \text{Im}[w] > 0\}.$$

We recall the semicircle distribution $\varrho_{\text{sc}}(x)$ and its Stieltjes transform $m_{\text{sc}}(z)$:

$$\varrho_{\text{sc}}(x) = \mathbf{1}_{x \in [-2, 2]} \frac{\sqrt{4-x^2}}{2\pi}, \quad m_{\text{sc}}(z) = \int_{\mathbb{R}} \frac{\varrho_{\text{sc}}(x) dx}{x - z} = \frac{-z + \sqrt{z^2 - 4}}{2}. \quad (1.5) \quad \{\text{e:msc_equat}\}$$

Explicitly, the Stieltjes transform of the Kesten-McKay distribution $m_d(z)$ can be expressed in terms of the Stieltjes transform $m_{\text{sc}}(z)$:

$$m_d(z) = \frac{1}{-z - \frac{d}{d-1} m_{\text{sc}}(z)}, \quad m_d(z) = (d-1) \frac{-(d-2)z + d\sqrt{z^2 - 4}}{2(d^2 - (d-1)z^2)}. \quad (1.6) \quad \{\text{e:md_equati}\}$$

1.2 Main Results

Our main result verifies the edge universality conjecture for random d -regular graphs by Sarnak [24] and Miller, Novikoff and Sabelli [23].

Theorem 1.1 (Edge eigenvalue universality [15]). *Fix $d \geq 3$, $k \geq 1$ and $s_1, s_2, \dots, s_k \in \mathbb{R}$, and let $\mathcal{A} = d(d-1)/(d-2)^2$ from (1.4). Then, for all $\varepsilon > 0$ such that the eigenvalues $\lambda_1 = d/\sqrt{d-1} \geq \lambda_2 \geq \dots \geq \lambda_N$ of the normalized adjacency matrix H of random d -regular graphs satisfy:*

$$\begin{aligned} & \mathbb{P}_H \left((\mathcal{A}N)^{2/3}(\lambda_2 - 2) \geq s_1, \dots, (\mathcal{A}N)^{2/3}(\lambda_{k+1} - 2) \geq s_k \right) \\ &= \mathbb{P}_{\text{GOE}} \left(N^{2/3}(\mu_1 - 2) \geq s_1, \dots, N^{2/3}(\mu_k - 2) \geq s_k \right) + O(N^{-\varepsilon}), \end{aligned}$$

where $\mu_1 \geq \mu_2 \geq \dots \geq \mu_N$ are the eigenvalues of the GOE. The analogous statement holds for the smallest eigenvalues $-\lambda_N, \dots, -\lambda_{N-k+1}$.

When the degree d grows with the size of the graph, edge universality for random d -regular graphs has been established previously for $N^{2/3+o(1)} \leq d \leq N/2$, by He [14], and for $N^{o(1)} \leq d \leq N^{1/3-o(1)}$, by the first and third authors of this paper [16], which generalized a result for $N^{2/9+o(1)} \leq d \leq N^{1/3-o(1)}$, by Bauerschmidt, Knowles and the first and third authors of this paper [2].

Theorem 1.1 implies that the fluctuation of the second largest eigenvalue converges to the Tracy-Widom₁ distribution. The Tracy-Widom₁ distribution has about 83% of its mass on the set $\{x : x < 0\}$ [11]. Therefore Theorem 1.1 implies 83% of d -regular graphs have the second eigenvalue *less than* 2. The proof of Theorem 1.1 can be extended to show that the largest and smallest nontrivial eigenvalues converge in distribution to *independent* Tracy-Widom₁ distributions. As a consequence, we have the following result. {c:rate}

Corollary 1.2 ([15]). *Fix $d \geq 3$ and N sufficiently large. With probability approximately 69%, a randomly sampled d -regular graph has $\max\{\lambda_2, |\lambda_N|\} \leq 2$, and is therefore Ramanujan.*

1.3 Parameters

In this note we fix the parameters as follows

$$0 < \mathfrak{o} \ll \mathfrak{b} \ll \mathfrak{c} \ll \mathfrak{g} \ll 1, \quad (1.7) \quad \{\mathfrak{e}:\text{parameter}\}$$

set $\mathfrak{R} = (\mathfrak{c}/4) \log_{d-1} N$ and choose ℓ such that $\ell / \log_{d-1} N \ll \mathfrak{b}$. Below, we describe their meanings and where they are introduced:

- Many estimates involve bounds containing $N^{\mathfrak{o}}$ factors, which are harmless.
- $\mathfrak{b}$ relates to the concentration of Green's function entries, with errors bounded by $N^{-\mathfrak{b}}$, see (3.20) and (3.21). {d:parameters}
- For the spectral parameter $z \in \mathbb{C}^+$ in Green's functions and Stieltjes transforms, we restrict it to $\text{Im}[z] \geq N^{-1+\mathfrak{g}}$, see (1.8).
- ℓ comes from local resampling in Section 5, we resample boundary edges of balls with radius ℓ .
- $\mathfrak{c}$ defines $\mathfrak{R}$, and with high probability, random d -regular graphs are tree-like within radius $\mathfrak{R}$ neighborhoods, see Definition 3.1.

We restrict our analysis to the spectral domain

$$\mathbf{D} = \{z \in \mathbb{C}^+ : N^{-1+\mathfrak{g}} \leq \text{Im}[z] \leq N^{-\mathfrak{o}}, |\text{Re}[z]| \leq 2 + N^{-\mathfrak{o}}\}. \quad (1.8) \quad \{\mathfrak{e}:\mathbf{D}\}$$

In the spectral domain $\mathbf{D}$, we impose the conditions $\text{Im}[z] \leq N^{-\mathfrak{o}}$ and $|\text{Re}[z]| \leq 2 + N^{-\mathfrak{o}}$. These constraints ensure that $|m_{\text{sc}}(z)|$ is close to 1, specifically satisfying $||m_{\text{sc}}(z)| - 1| \lesssim N^{-\mathfrak{o}/2}$.

2 Properties of the Green's functions

{s:Greenf}

Throughout this paper, we repeatedly use some (well-known) identities for Green's functions, which we collect them here.

2.1 Resolvent identity

The following well-known identity is referred as resolvent identity: for two invertible matrices A and B of the same size, we have

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} = B^{-1}(B - A)A^{-1}. \quad (2.1) \quad \{\mathfrak{e}:\text{resolv}\}$$

2.2 Woodbury formula

Let $A + UCV^\top$ be a rank r perturbation of A . Namely, $U, V \in \mathbb{R}^{N \times r}$ and $C \in \mathbb{R}^{r \times r}$. Then, the Woodbury formula gives us

$$(A + UCV^\top)^{-1} - A^{-1} = -A^{-1}U(C^{-1} + V^\top A^{-1}U)^{-1}V^\top A^{-1}. \quad (2.2) \quad \{\mathbf{e:woodbury}\}$$

2.3 Schur complement formula

Given an $N \times N$ matrix H and an index set $\mathbb{T} \subset \llbracket N \rrbracket$, recall that we denote by $H|_{\mathbb{T}}$ the $\mathbb{T} \times \mathbb{T}$ -matrix obtained by restricting H to $\mathbb{T}$, and that by $H^{(\mathbb{T})} = H|_{\mathbb{T}^c}$ the matrix obtained by removing the rows and columns corresponding to indices in $\mathbb{T}$. Thus, for any $\mathbb{T} \subset \llbracket N \rrbracket$, any symmetric matrix H can be written (up to rearrangement of indices) in the block form

$$H = \begin{bmatrix} A & B^\top \\ B & D \end{bmatrix},$$

with $A = H|_{\mathbb{T}}$ and $D = H^{(\mathbb{T})}$. The Schur complement formula asserts that, for any $z \in \mathbb{C}^+$,

$$G = (H - z)^{-1} = \begin{bmatrix} (A - B^\top G^{(\mathbb{T})} B)^{-1} & -(A - B^\top G^{(\mathbb{T})} B)^{-1} B^\top G^{(\mathbb{T})} \\ -G^{(\mathbb{T})} B (A - B^\top G^{(\mathbb{T})} B)^{-1} & G^{(\mathbb{T})} + G^{(\mathbb{T})} B (A - B^\top G^{(\mathbb{T})} B)^{-1} B^\top G^{(\mathbb{T})} \end{bmatrix}, \quad (2.3) \quad \{\mathbf{e:Schur}\}$$

where $G^{(\mathbb{T})} = (D - z)^{-1}$. Throughout the paper, we often use the following special cases of (2.3): $\{\mathbf{e:Schur}\}$

$$\begin{aligned} G|_{\mathbb{T}} &= (A - B^\top G^{(\mathbb{T})} B)^{-1}, \\ G|_{\mathbb{T}\mathbb{T}^c} &= -G|_{\mathbb{T}} B^\top G^{(\mathbb{T})}, \\ G|_{\mathbb{T}^c} &= G^{(\mathbb{T})} + G|_{\mathbb{T}\mathbb{T}^c} (G|_{\mathbb{T}})^{-1} G|_{\mathbb{T}\mathbb{T}^c} = G^{(\mathbb{T})} - G^{(\mathbb{T})} B G|_{\mathbb{T}\mathbb{T}^c}, \end{aligned} \quad (2.4) \quad \{\mathbf{e:Schur1}\}$$

as well as the special case

$$G_{ij}^{(k)} = G_{ij} - \frac{G_{ik} G_{kj}}{G_{kk}} = G_{ij} + (G^{(k)} H)_{ik} G_{kj}. \quad (2.5) \quad \{\mathbf{e:Schurixj}\}$$

2.4 Ward identity

For any symmetric $N \times N$ matrix H , its Green's function $G(z) = (H - z)^{-1}$ satisfies the Ward identity

$$\sum_{j=1}^N |G_{ij}(z)|^2 = \frac{\text{Im } G_{jj}(z)}{\eta}, \quad (2.6) \quad \{\mathbf{e:Ward}\}$$

where $\eta = \text{Im}[z]$. This can be deduced from using (2.1) on $G \mathbf{e} \mathbf{e}^* G^*$. This identity provides a bound for the sum $\sum_{j=1}^N |G_{ij}(z)|^2$ in terms of the diagonal entries of the Green's function.

3 Local law for random d -regular graphs

$\{\mathbf{s:local_law}\}$

In this section, we present the local law results from [17]. It states that the Green's function of random d -regular graphs are approximated by the Green's function extension with general weights, which are easy to compute. We introduce the concept of Green's function extension with general weights in Section 3.2, and state the local law in Section 3.

3.1 Locally tree-like graphs

Random d -regular graphs are locally tree-like in the sense that each radius $O(\log_{d-1}(N))$ neighborhoods is either a truncated tree (contains no cycles), or contains at most one cycle.

$\{\mathbf{def:omegabar}\}$

Definition 3.1. Fix $d \geq 3$ and a sufficiently small $0 < \mathfrak{c} < 1$, $\mathfrak{R} = (\mathfrak{c}/4) \log_{d-1} N$ as in (1.7). We define the event $\overline{\Omega}$, where the following occur:

1. The number of vertices that do not have a tree neighborhood of radius $\mathfrak{R}$ is at most N^ϵ .
2. The radius $\mathfrak{R}$ neighborhood of each vertex has an excess (i.e., the number of independent cycles) of at most 1.

The event $\bar{\Omega}$ is a typical event. The following proposition from [17, Proposition 2.1] states that $\bar{\Omega}$ holds with high probability. {lem:omega}

Proposition 3.2 ([17, Proposition 2.1]). $\bar{\Omega}$ occurs with probability $1 - O(N^{-(1-\epsilon)})$.

As we will see, for graphs $\mathcal{G} \in \bar{\Omega}$, their Green's functions can be approximated by tree extensions with overwhelmingly high probability. For the infinite d -regular tree and the infinite $(d-1)$ -ary tree (trees where the root has degree $d-1$ and all other vertices have degree d), the following proposition computes their Green's function explicitly. {greentree}

Proposition 3.3 ([17, Proposition 2.2]). Let $\mathcal{X}$ be the infinite d -regular tree. For all $z \in \mathbb{C}^+$, its Green's function is

$$G_{ij}(z) = m_d(z) \left(-\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{X}}(i,j)}. \quad (3.1) \quad \{\text{e:Gtreekm}\}$$

Let $\mathcal{Y}$ be the infinite $(d-1)$ -ary tree with root vertex o . Its Green's function is

$$G_{ij}(z) = m_d(z) \left(1 - \left(-\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{2\text{anc}(i,j)+2} \right) \left(-\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{Y}}(i,j)}, \quad (3.2) \quad \{\text{e:Gtreemsc}\}$$

where $\text{anc}(i, j)$ is the distance from the common ancestor of the vertices i, j to the root o . In particular,

$$G_{oi}(z) = m_{\text{sc}}(z) \left(-\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{Y}}(o,i)}. \quad (3.3) \quad \{\text{e:Gtreemsc2}\}$$

3.2 Green's function extension with general weights {s:pre}

We recall the notion of a Green's function extension with general weight Δ from [17, Section 2.3]. Roughly speaking, we consider the radius- ℓ neighborhood $\mathcal{T} = \mathcal{B}_\ell(o, \mathcal{G})$ of a vertex o in $\mathcal{G}$. Assume it is a tree, then the boundary vertices of $\mathcal{T}$ are leaves (i.e., vertices of degree 1), to each of which we attach a weight $-\Delta$; see the left panel of Figure 1. The resulting Green's function of $\mathcal{T}$ with these boundary weights is referred to as the Green's function extension with general weight Δ . The following formal definition slightly extend this idea to allow cycles. {def:pdef}

Definition 3.4. Fix degree $d \geq 3$, and a graph $\mathcal{T}$ with degrees bounded by d . We define the function $P(\mathcal{T}, z, \Delta)$ as follows. We denote $A(\mathcal{T})$ the adjacency matrix of $\mathcal{T}$, $D(\mathcal{T})$ the diagonal matrix of degrees of $\mathcal{T}$, and $\mathbb{I}$ the diagonal matrix indexed by the vertex set $\mathbb{T}$ of $\mathcal{T}$. Then

$$P(\mathcal{T}, z, \Delta) := \frac{1}{-z + A(\mathcal{T})/\sqrt{d-1} - (d\mathbb{I} - D(\mathcal{T}))\Delta/(d-1)}. \quad (3.4) \quad \{\text{e:defP}\}$$

The matrix $P(\mathcal{T}, z, \Delta)$ is the Green's function of the matrix obtained from $A(\mathcal{T})/\sqrt{d-1}$ by attaching to each vertex $i \in \mathcal{T}$ a weight $-(d - D_{ii}(\mathcal{T}))\Delta/(d-1)$, see the middle panel of Figure 1. When $\Delta = m_{\text{sc}}(z)$, (3.4) is the Green's function of the tree extension of $\mathcal{T}$, i.e. extending $\mathcal{T}$ by attaching copies of infinite $(d-1)$ -ary trees to $\mathcal{T}$ to make each vertex degree d . If $\mathcal{T}$ is a tree, then in this case, the Green's function agrees with the Green's function of the infinite d -regular tree, as in (3.1). For any vertex set $\mathbb{X}$ in $\mathcal{T}$, we define the following Green's function with vertex $\mathbb{X}$ removed, see the right panel of Figure 1. Let $A^{(\mathbb{X})}(\mathcal{T})$ and $D^{(\mathbb{X})}(\mathcal{T})$ denote the matrices obtained from $A(\mathcal{T}), D(\mathcal{T})$ by removing the row and column associated with vertex $\mathbb{X}$, and $\mathbb{I}$ the diagonal matrix indexed by the vertex set $\mathbb{T} \setminus \mathbb{X}$. The Green's function is then defined as:

$$P^{(\mathbb{X})}(\mathcal{T}, z, \Delta) := \frac{1}{-z + A^{(\mathbb{X})}(\mathcal{T})/\sqrt{d-1} - (d\mathbb{I} - D^{(\mathbb{X})}(\mathcal{T}))\Delta/(d-1)}, \quad (3.5) \quad \{\text{e:defPi}\}$$

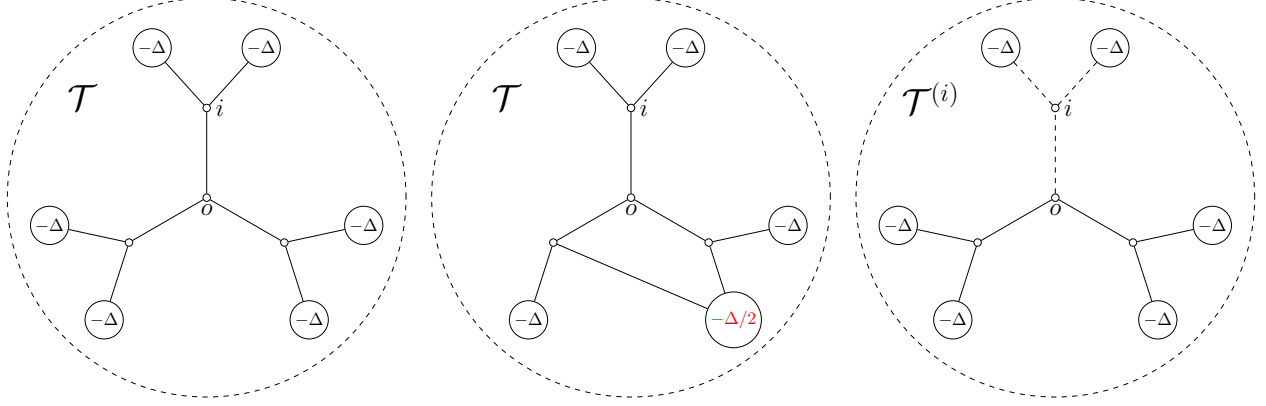


Figure 1: Left panel: for a truncated d -regular tree $\mathcal{T}$, we attach a weight $-\Delta$ to each leaf vertex; middle panel: for a general graph $\mathcal{T}$, we attach a weight $-(d - D_{ii}(\mathcal{T}))/ (d - 1)$ to each vertex; right panel: remove the vertex set $\mathbb{X} = \{i\}$.

{fig:three-tr

For any integer $\ell \geq 1$, we define the functions $X_\ell(\Delta, z), Y_\ell(\Delta, z)$ as

$$X_\ell(\Delta, z) = P_{oo}(\mathcal{B}_\ell(o, \mathcal{X}), z, \Delta), \quad Y_\ell(\Delta, z) = P_{oo}(\mathcal{B}_\ell(o, \mathcal{Y}), z, \Delta), \quad (3.6) \quad \{\text{def:Y}\}$$

where $\mathcal{X}$ is the infinite d -regular tree with root vertex o , and $\mathcal{Y}$ is the infinite $(d-1)$ -ary tree with root vertex o . Then $m_{\text{sc}}(z)$ is a fixed point of the function Y_ℓ , i.e. $Y_\ell(m_{\text{sc}}(z), z) = m_{\text{sc}}(z)$. Also, $X_\ell(m_{\text{sc}}(z), z) = m_d(z)$. The following proposition states that if Δ is sufficiently close to $m_{\text{sc}}(z)$ and w is close to z , then $Y_\ell(\Delta, w)$ is close to $m_{\text{sc}}(z)$, and $X_\ell(\Delta, w)$ is close to $m_d(z)$.

{p:recurbound

Proposition 3.5. *Given $z, \Delta, \Delta' \in \mathbb{C}^+$ such that $\ell^2|\Delta - m_{\text{sc}}(z)|, \ell^2|\Delta' - m_{\text{sc}}(z)| \ll 1$, then the functions $X_\ell(\Delta, z), Y_\ell(\Delta, z)$ satisfy*

$$\begin{aligned} Y_\ell(\Delta, z) - m_{\text{sc}}(z) &= m_{\text{sc}}^{2\ell+2}(z) m_d(z) \left(\frac{1 - m_{\text{sc}}^{2\ell+2}(z)}{d-1} + \frac{d-2}{d-1} \frac{1 - m_{\text{sc}}^{2\ell+2}(z)}{1 - m_{\text{sc}}^2(z)} \right) (\Delta - m_{\text{sc}}(z))^2 \\ &\quad + m_{\text{sc}}^{2\ell+2}(z) (\Delta - m_{\text{sc}}(z)) + O(\ell^5 |\Delta - m_{\text{sc}}(z)|^3). \end{aligned} \quad (3.7) \quad \{\text{e:recurbound}$$

and

$$X_\ell(\Delta, z) - m_d(z) = \frac{d}{d-1} m_d^2(z) m_{\text{sc}}^{2\ell}(z) (\Delta - m_{\text{sc}}(z)) + O(|\Delta - m_{\text{sc}}(z)|^2). \quad (3.8) \quad \{\text{e:Xrecurbound}$$

Moreover, the difference $Y_\ell(\Delta, z) - Y_\ell(\Delta', z)$ satisfies

$$|Y_\ell(\Delta, z) - Y_\ell(\Delta', z)| \lesssim \ell |\Delta - \Delta'|, \quad (3.9) \quad \{\text{e:Yl_deriva}$$

Proof of Proposition 3.5. The proofs for X_ℓ and Y_ℓ are identical, so we will only provide the proof for Y_ℓ . We denote $\mathcal{H} = \mathcal{B}_\ell(o, \mathcal{Y})$, which is the truncated $(d-1)$ -ary tree at level ℓ . We denote its vertex set as $\mathbb{H}$ and normalized adjacency matrix as H . We denote $\mathbb{I}$ and $\mathbb{I}^\partial$ the diagonal matrices, such that for $x, y \in \mathbb{H}$, $\mathbb{I}_{xy} = \delta_{xy}$ and $\mathbb{I}_{xy}^\partial = \mathbf{1}(\text{dist}_{\mathcal{H}}(x, o) = \ell) \delta_{xy}$. Then (3.4) gives the derivative

$$Y_\ell(\Delta, z) = P(\mathcal{H}, z, \Delta) = (H - z - \Delta \mathbb{I}^\partial)^{-1}, \quad P = P(\mathcal{H}, z, m_{\text{sc}}(z)) = (H - z - m_{\text{sc}}(z) \mathbb{I}^\partial)^{-1}.$$

In the rest of the proof, we will simply write $m_{\text{sc}} = m_{\text{sc}}(z)$. We can compute the Green's function $P(\mathcal{H}, z, \Delta)$ by a perturbation argument,

$$\begin{aligned} P(\mathcal{H}, z, \Delta) &= (H - z - \Delta \mathbb{I}^\partial)^{-1} = (H - z - m_{\text{sc}} \mathbb{I}^\partial + (\Delta - m_{\text{sc}}) \mathbb{I}^\partial)^{-1} \\ &= P + \sum_{k \geq 1} (\Delta - m_{\text{sc}})^k P (\mathbb{I}^\partial P)^k. \end{aligned} \quad (3.10) \quad \{\text{e:expansion}$$

With the explicit expression of P as given in (3.3), we can compute

$$(P\mathbb{I}^\partial P)_{oo} = \sum_l P_{ol} P_{lo} = \sum_{l,l'} m_{\text{sc}}^2 \left(-\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2\ell} = m_{\text{sc}}^{2\ell+2} \quad \{\mathbf{e:PBPool}\} \quad (3.11)$$

where the summation is over l, l' such that $\text{dist}_{\mathcal{H}}(l, o) = \text{dist}_{\mathcal{H}}(l', o) = \ell$, and there are $(d-1)^\ell$ values of l . Moreover, for $k \geq 2$ we will show the following two relations for P ,

$$(P\mathbb{I}^\partial P\mathbb{I}^\partial P)_{oo} = m_{\text{sc}}^{2\ell+2} m_d \left(\frac{1 - m_{\text{sc}}^{2\ell+2}}{d-1} + \frac{d-2}{d-1} \frac{1 - m_{\text{sc}}^{2\ell+2}}{1 - m_{\text{sc}}^2} \right), \quad \{\mathbf{e:Pboundary}\} \quad (3.12)$$

$$(|P|^k)_{oo} \lesssim (C\ell)^{2k-3}, \quad \{\mathbf{e:Ptotalsum}\} \quad (3.13)$$

where for each $i, j \in \mathbb{H}$, $|P|_{ij} := |P_{ij}|$.

The relation (3.12) follows from an explicit computation using (3.2) and (3.3)

$$\begin{aligned} \sum_{l,l'} P_{ol} P_{ll'} P_{l'o} &= \sum_{l,l'} m_{\text{sc}}^2 \left(-\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2\ell} m_d \left(1 - \left(-\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2+2\text{anc}(l,l')} \right) \left(-\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{H}}(l,l')} \\ &= \frac{m_{\text{sc}}^{2\ell+2}}{(d-1)^{2\ell}} \sum_{l,l'} m_d \left(1 - \left(\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2+2\text{anc}(l,l')} \right) \left(-\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{H}}(l,l')} \\ &= m_{\text{sc}}^{2\ell+2} m_d \left(1 - \left(\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2+2\ell} + \sum_{r=1}^{\ell} \left(1 - \left(\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2+2(\ell-r)} \right) \left(\frac{m_{\text{sc}}}{\sqrt{d-1}} \right)^{2r} (d-2)(d-1)^{r-1} \right) \\ &= m_{\text{sc}}^{2\ell+2} m_d \left(\frac{1 - m_{\text{sc}}^{2\ell+2}}{d-1} + \frac{d-2}{d-1} \frac{1 - m_{\text{sc}}^{2\ell+2}}{1 - m_{\text{sc}}^2} \right), \end{aligned}$$

where the summation in the first line is over l, l' such that $\text{dist}_{\mathcal{H}}(l, o) = \text{dist}_{\mathcal{H}}(l', o) = \ell$; in the second to last line we used that for a given l , there are $(d-2)(d-1)^{r-1}$ values of l' such that $\text{dist}_{\mathcal{H}}(l, l') = 2r$, $\text{anc}(l, l') = \ell - r$ for $1 \leq r \leq \ell$. When $l = l'$, it holds $\text{dist}_{\mathcal{H}}(l, l') = 0$, $\text{anc}(l, l') = \ell$.

To prove (3.13), we show by induction that for any vertex i such that $\text{dist}_{\mathcal{H}}(o, i) \leq \ell$,

$$(|P|^k)_{oi} \lesssim \frac{(C\ell)^{2k-2}}{(d-1)^{\text{dist}_{\mathcal{H}}(o,i)/2}}. \quad \{\mathbf{e:induction}\} \quad (3.14)$$

The statement for $k = 1$ follows from (3.2). Assuming the statement (3.14) holds for $k-1$, we prove it for k ,

$$(|P|^k)_{oi} = \sum_{j \in \mathbb{H}} (|P|^{k-1})_{oj} |P|_{ji} \lesssim \sum_{j \in \mathbb{H}} \frac{(C\ell)^{2k-4}}{(d-1)^{\text{dist}_{\mathcal{H}}(o,j)/2}} \frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(j,i)/2}}. \quad \{\mathbf{e:induction}\} \quad (3.15)$$

We denote the path from o to i as $\mathcal{P}$, then $|\{j \in \mathbb{H} : \text{dist}_{\mathcal{H}}(j, \mathcal{P}) = r\}| \leq \ell(d-1)^r$, and for $\text{dist}_{\mathcal{H}}(j, \mathcal{P}) = r$, we have

$$\frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(o,j)/2}} \frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(j,i)/2}} \lesssim \frac{1}{(d-1)^r} \frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(o,i)/2}}.$$

In this way, we can reorganize the sum over j in (3.15) according to its distance to $\mathcal{P}$,

$$\begin{aligned} (|P|^k)_{oi} &= \sum_{r=0}^{\ell} \sum_{j \in \mathbb{H} : \text{dist}_{\mathcal{H}}(j, \mathcal{P})=r} \frac{(C\ell)^{2k-4}}{(d-1)^{\text{dist}_{\mathcal{H}}(o,j)/2}} \frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(j,i)/2}} \\ &\lesssim \sum_{r=0}^{\ell} |\{j : \text{dist}_{\mathcal{H}}(j, \mathcal{P}) = r\}| \frac{(C\ell)^{2k-4}}{(d-1)^r} \frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(o,i)/2}} \lesssim \frac{(C\ell)^{2k-2}}{(d-1)^{\text{dist}_{\mathcal{H}}(o,i)/2}}, \end{aligned}$$

which shows (3.14). The claim (3.13) is a consequence of (3.14)

$$(|P|^k)_{oo} = \sum_{i \in \mathbb{H}} (|P|^{k-1})_{oi} |P|_{io} \lesssim \sum_{i \in \mathbb{H}} \frac{(C\ell)^{2k-4}}{(d-1)^{\text{dist}_{\mathcal{H}}(o,i)/2}} \frac{1}{(d-1)^{\text{dist}_{\mathcal{H}}(i,o)/2}} \lesssim (C\ell)^{2k-3}.$$

The claim (3.7) follows from plugging (3.11), (3.12) and (3.13) into (3.10), and use $\ell^2 |\Delta - m_{\text{sc}}| \ll 1$:

$$\begin{aligned} Y_\ell(\Delta, z) - m_{\text{sc}}(z) &= \sum_{k \geq 1} (\Delta - m_{\text{sc}})^k (P(\mathbb{I}^\partial P)^k)_{oo} \\ &= m_{\text{sc}}^{2\ell+2}(z) m_d(z) \left(\frac{1 - m_{\text{sc}}^{2\ell+2}(z)}{d-1} + \frac{d-2}{d-1} \frac{1 - m_{\text{sc}}^{2\ell+2}(z)}{1 - m_{\text{sc}}^2(z)} \right) (\Delta - m_{\text{sc}}(z))^2 \\ &\quad + m_{\text{sc}}^{2\ell+2}(z) (\Delta - m_{\text{sc}}(z)) + \sum_{k \geq 3} |\Delta - m_{\text{sc}}(z)|^k (|P|^{k+1})_{oo}. \end{aligned}$$

The claim (3.9) follows from bounding the derivative

$$|\partial_1 Y_\ell(\Delta, z)| \lesssim \ell. \quad (3.16) \quad \{\text{fe:derivative}\}$$

The claim (3.16) follows from taking derivative with respect to Δ on both sides of (3.10), and plugging in (3.11) and (3.13)

$$|\partial_1 Y_\ell(\Delta, z)| \lesssim |(P\mathbb{I}^\partial P)_{oo}| + \sum_{k \geq 1} |\Delta - m_{\text{sc}}|^k (|P|^{k+2})_{oo} \lesssim 1 + C\ell \sum_{k \geq 1} (C\ell^2 |\Delta - m_{\text{sc}}|)^k \lesssim \ell. \quad (3.17)$$

□

3.3 Local law of H

To state the local law of H , we recall $Q(z)$, a quantity that was first defined in [3, 17], and plays a crucial role in the proof of local law. The quantity is the average of $G_{jj}^{(i)}(z)$ over all pairs of adjacent vertices $i \sim j$:

$$Q(z) := \frac{1}{Nd} \sum_{i \sim j} G_{jj}^{(i)}(z). \quad (3.18) \quad \{\text{fe:Qsum}\}$$

For any vertex set $\mathbb{X} \subset \llbracket N \rrbracket$, and integer $r \geq 1$, we denote $\mathcal{B}_r(\mathbb{X}, \mathcal{G}) = \{i \in \llbracket N \rrbracket : \text{dist}_{\mathcal{G}}(i, \mathbb{X}) \leq r\}$ the ball of radius- r around vertices $\mathbb{X}$ in $\mathcal{G}$. The weak local law of $H = H(0)$ has been proven in [17, Theorem 4.2], which is recalled below (by taking $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{r}) = (12, 300, \mathfrak{c}, \mathfrak{c}/32)$). \{thm:prevthm0\}

Theorem 3.6 ([17, Theorem 4.2]). *Fix any sufficiently small $0 < \mathfrak{b} \ll \mathfrak{c} < 1$, $\mathfrak{R} = (\mathfrak{c}/4) \log_{d-1}(N)$ and any $z \in \mathbb{C}^+$, we define $\eta(z) = \text{Im}[z]$, $\kappa(z) = \min\{|\text{Re}[z] - 2|, |\text{Re}[z] + 2|\}$, and the error parameters*

$$\varepsilon'(z) := (\log N)^{100} \left(N^{-10\mathfrak{b}} + \sqrt{\frac{\text{Im}[m_d(z)]}{N\eta(z)}} + \frac{1}{(N\eta(z))^{2/3}} \right), \quad \varepsilon(z) := \frac{\varepsilon'(z)}{\sqrt{\kappa(z) + \eta(z) + \varepsilon'(z)}}. \quad (3.19) \quad \{\text{fe:defepsilo}\}$$

For any $\mathfrak{C} \geq 1$ and N large enough, with probability at least $1 - O(N^{-\mathfrak{C}})$ with respect to the uniform measure on $\overline{\Omega}$,

$$|G_{ij}(z) - P_{ij}(\mathcal{B}_{\mathfrak{R}/100}(\{i, j\}, \mathcal{G}), z, m_{\text{sc}}(z))|, \quad |Q(z) - m_{\text{sc}}(z)|, \quad |m_N(z) - m_d(z)| \lesssim \varepsilon(z). \quad (3.20) \quad \{\text{feq:infbound}\}$$

uniformly for every $i, j \in \llbracket N \rrbracket$, and any $z \in \mathbb{C}^+$ with $\text{Im}[z] \geq (\log N)^{300}/N$. \{\text{def:omega}\}

Definition 3.7. We denote by $\Omega \subset \overline{\Omega}$ the set of d -regular graphs for which (3.20) holds. \{\text{def:infbound0}\}

As a consequence of Theorem 3.6 we have the following statement.

Claim 3.8. *Uniformly for every $i, j \in \llbracket N \rrbracket$, and any $z \in \mathbb{C}^+$ with $|z| \leq 1/\mathfrak{g}$, $\text{Im}[z] \geq N^{-1+\mathfrak{g}}$, we have the following bounds:*

$$\begin{aligned} |G_{ij}(z) - P_{ij}(\mathcal{B}_{\mathfrak{R}/100}(\{i, j\}, \mathcal{G}), z, m_{\text{sc}}(z))| &\lesssim N^{-2\mathfrak{b}}, \\ |Q(z) - m_{\text{sc}}(z)|, \quad |m_N(z) - m_d(z)| &\lesssim N^{-2\mathfrak{b}}. \end{aligned} \quad (3.21) \quad \{\text{eq:infbound}\}$$

and for any $i, j \in \llbracket N \rrbracket$

$$\text{Im}[G_{ij}] \lesssim N^0 \text{Im}[m_N(z)] \quad (3.22) \quad \{\text{e:Gsize}\}$$

Proof. For $\text{Im}[z] \geq N^{-1+\mathfrak{g}}$, thanks to (3.19), $\{\text{e:Gsize}\}$

$$\varepsilon(z) \leq \sqrt{\varepsilon'(z)} \leq N^{-4\mathfrak{b}}. \quad (3.23) \quad \{\text{e:epsilon_b}\}$$

Thus, the claim (3.21) follows from (3.20).

We denote the eigenvectors of H as $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_N$. We notice that the first statement in (3.20) implies that for $\text{Im}[z] \geq (\log N)^{300}/N$, $\text{Im}[G_{ii}(z)] \lesssim 1$. It follows that eigenvectors are delocalized

$$\|\mathbf{u}_\alpha\|_\infty^2 \lesssim \max_{1 \leq i \leq N} ((\log N)^{300}/N) \text{Im}[G_{ii}(\lambda_\alpha + i(\log N)^{300}/N)] \lesssim (\log N)^{300}/N \ll N^{0/2-1}.$$

Thus (3.22) follows

$$|\text{Im}[G_{xy}(z)]| = \left| \sum_{\alpha=1}^N \frac{\eta(\mathbf{u}_\alpha \mathbf{u}_\alpha^\top)_{xy}}{|\lambda_\alpha - z|^2} \right| \leq N^{0/2-1} \sum_{\alpha=1}^N \frac{\eta}{|\lambda_\alpha - z|^2} = N^{0/2} \text{Im}[m_N(z)]. \quad (3.24) \quad \{\text{e:smallGij}\}$$

□ $\{\text{l:basicG}\}$

Lemma 3.9. *Let $z \in \mathbb{C}^+$ satisfy $|z| \leq 1/\mathfrak{g}$, and set $\eta := \text{Im } z \geq N^{-1+\mathfrak{g}}$. Fix a d -regular graph $\mathcal{G} \in \Omega$ (recall Definition 3.7) and vertices $\{i, o\}$ in $\mathcal{G}$, and let $\mathcal{T} = \mathcal{B}_\ell(o, \mathcal{G})$ with vertex set $\mathbb{T}$. For any $0 \leq \mu \lesssim (d-1)^\ell$, take a family of directed edges $\{(b_\alpha, c_\alpha)\}_{\alpha \in \llbracket \mu \rrbracket}$, set $\mathbb{W} = \{b_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}$, and condition on the event that*

$$A_{io} \prod_{\alpha \in \llbracket \mu \rrbracket} A_{b_\alpha c_\alpha} \prod_{x \neq y \in \{o, c_1, c_2, \dots, c_\mu\}} \mathbf{1}(\mathcal{B}_{\mathfrak{R}/2}(x, \mathcal{G}) \text{ is a tree}) \mathbf{1}(\text{dist}_{\mathcal{G}}(x, y) \geq \mathfrak{R}/2) = 1. \quad (3.25) \quad \{\text{e:condition}\}$$

Then, for any $\mathbb{X}$ of the form $\mathbb{X} \in \{\mathbb{W}, \mathbb{T} \cup \mathbb{W}\}$ and any $x, y \notin \mathbb{X}$, we have

$$|G_{xy}^{(\mathbb{X})}(z) - P_{xy}^{(\mathbb{X})}(\mathcal{B}_{\mathfrak{R}/100}(\{x, y\} \cup \mathbb{X}, \mathcal{G}), z, m_{\text{sc}}(z))| \lesssim N^{-\mathfrak{b}}, \quad |\text{Im } G_{xy}^{(\mathbb{X})}(z)| \lesssim N^0 \text{Im } m_N(z), \quad (3.26) \quad \{\text{eq:local_la}\}$$

and for any $x \notin \mathbb{X}$,

$$\frac{1}{N} \sum_{b \sim c \notin \mathbb{X}} |G_{cx}^{(\mathbb{X})}(z)|^2 \lesssim \frac{N^0 \text{Im } m_N(z)}{N\eta}, \quad \frac{1}{Nd} \sum_{b \sim c \notin \mathbb{X}} |G_{cx}^{(b\mathbb{X})}(z)|^2 \lesssim \frac{N^0 \text{Im } m_N(z)}{N\eta}. \quad (3.27) \quad \{\text{e:Gest}\}$$

Proof. We will only prove the case that $\mathbb{X} = \mathbb{T}$, the other cases can be proven in the same way, so we omit. From the Schur complement formula (2.4), $\{\text{e:Gest}\}$

$$G^{(\mathbb{T})} = G - G(G|_{\mathbb{T}})^{-1}G. \quad (3.28) \quad \{\text{e:GT}\}$$

We denote $\mathcal{T} = \mathcal{B}_\ell(o, \mathcal{G})$ and $P = P(\mathcal{B}_{\mathfrak{R}/100}(\mathbb{T} \cup \{x, y\}, \mathcal{G}), z, m_{\text{sc}}(z))$. Then (3.4) gives $G|_{\mathbb{T}}^{-1}|_{\mathbb{T}} = H|_{\mathbb{T}} - z$, where the diagonal matrix $\mathbb{I}_{ij}^\partial = \mathbf{1}(\text{dist}_{\mathcal{T}}(o, i) = \ell) \delta_{ij}$ for $i, j \in \mathbb{T}$. It follows that for any $i \in \mathbb{T}$,

$$\sum_{j \in \mathbb{T}} |(P|_{\mathbb{T}})^{-1}|_{ij} \lesssim \sum_{j \in \mathbb{T}} |(H - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{ij}| \lesssim 1. \quad (3.29) \quad \{\text{e:sumPTSbd}\}$$

We recall that by (3.25) the event $\mathcal{E}_{\mathfrak{R}/50}$ is a tree neighborhood of radius $\mathfrak{R}/2$. When restricted to the connected component of o in $\mathcal{B}_{\mathfrak{R}/100}(\mathbb{T} \cup \{x, y\}, \mathcal{G})$, P is the Green's function of the d -regular tree. Thus, for any i, j in this connected component, (3.21) implies

$$|G_{ij} - P_{ij}| \leq N^{-2b}. \quad (3.30) \quad \{\mathbf{e}:\mathbf{G-P}\}$$

If x is not in the connected component of o in $\mathcal{B}_{\mathfrak{R}/100}(\mathbb{T} \cup \{x, y\}, \mathcal{G})$, then $\text{dist}_{\mathcal{G}}(x, j) > \mathfrak{R}/50$ for any $j \in \mathbb{T}$. In this case $P_{xj} = 0$, and by (3.21) we have $|G_{xj}| \leq N^{-2b}$. We also have

$$|G_{ij} - P_{ij}|, \quad |G_{xj} - P_{xj}| \leq N^{-2b}, \quad i, j \in \mathbb{T}. \quad (3.31) \quad \{\mathbf{e}:\mathbf{G-P1}\}$$

We can perform a resolvent expansion according to (2.1) and write $(G|_{\mathbb{T}})^{-1}_{ij}$ as: for some large enough $p \geq 1$

$$\begin{aligned} (G|_{\mathbb{T}})^{-1}_{ij} - P_{ij}^{-1} &= \left(\sum_{k=1}^p P^{-1} ((P - G|_{\mathbb{T}}) P^{-1})^k \right)_{ij} + O\left(\frac{1}{N}\right) \\ &\lesssim \sum_{l \in \mathbb{T}} |(P^{-1})_{il}| N^{-2b} \sum_{l \in \mathbb{T}} |(P^{-1})_{lj}| + O\left(\frac{1}{N}\right) \lesssim N^{-2b}, \end{aligned} \quad (3.32) \quad \{\mathbf{e}:\mathbf{sum_Ginve}\}$$

where in the second statement we used (3.29) and (3.31). From the above expression, we get

$$\sum_{j \in \mathbb{T}} |(G|_{\mathbb{T}})^{-1}_{ij}| \lesssim \sum_{j \in \mathbb{T}} |P_{ij}^{-1}| + O(|\mathbb{T}| N^{-2b}) \lesssim 1. \quad (3.33) \quad \{\mathbf{e}:\mathbf{sum_Ginve}\}$$

Again, since the vertex o has radius $\mathfrak{R}/2$ tree neighborhood, for any $0 \leq r \leq \ell$, $|\{j \in \mathbb{T} : \text{dist}_{\mathcal{G}}(x, j) = r\}| \lesssim (d-1)^r$. As a consequence, for any $i \in \mathbb{T}$, (3.31) and (3.31) together imply that

$$\begin{aligned} \sum_{j \in \mathbb{T}} |P_{xj}| &\lesssim \sum_{j \in \mathbb{T}} \left(\frac{|m_{\text{sc}}|}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{T}}(x, j)} \lesssim \sum_{0 \leq r \leq 2\ell} \frac{|\{j \in \mathbb{T} : \text{dist}(x, j) = r\}|}{(d-1)^{r/2}} \lesssim \ell, \\ \sum_{j \in \mathbb{T}} |G_{xj}| &\lesssim \sum_{j \in \mathbb{T}} |P_{xj}| + O(|\mathbb{T}| N^{-2b}) \lesssim \ell. \end{aligned} \quad (3.34) \quad \{\mathbf{e}:\mathbf{sumGix}\}$$

For P , Schur complement formula (2.4) gives $P^{(\mathbb{T})} = P - P(P|_{\mathbb{T}})^{-1}P$. By taking difference with (3.28), we get

$$\begin{aligned} |G_{xy}^{(\mathbb{T})} - P_{xy}^{(\mathbb{T})}| &\lesssim |G_{xy} - P_{xy}| + \sum_{i, j \in \mathbb{T}} |G_{xi} - P_{xi}| |(G|_{\mathbb{T}})^{-1}_{ij}| |G_{jy}| \\ &+ \sum_{i, j \in \mathbb{T}} |P_{xi}| |(G|_{\mathbb{T}})^{-1}_{ij} - (P|_{\mathbb{T}})^{-1}_{ij}| |G_{jy}| + \sum_{i, j \in \mathbb{T}} |P_{xi}| |(P|_{\mathbb{T}})^{-1}_{ij}| |G_{jy} - P_{jy}| \\ &\lesssim |G_{xy} - P_{xy}| + N^{-2b} \sum_{i, j \in \mathbb{T}} (|(G|_{\mathbb{T}})^{-1}_{ij}| |G_{jy}| + |P_{xi}| |G_{jy}| + |P_{xi}| |(P|_{\mathbb{T}})^{-1}_{ij}|) \\ &\lesssim \ell^2 N^{2b} \lesssim N^{-b}. \end{aligned} \quad (3.35) \quad \{\mathbf{e}:\mathbf{GTbbbb}\}$$

where in the second inequality, we used (3.31) and (3.33), in the third inequality we used (3.29), (3.34) and (3.33). This finishes the proof of the first statement in (3.26).

By taking imaginary part on both sides of (3.28), we get

$$\begin{aligned} \text{Im}[G_{xy}^{(\mathbb{T})}] &= \text{Im}[G_{xy}] - \text{Im}[(G(G|_{\mathbb{T}})^{-1}G)_{xy}] \lesssim \text{Im}[G_{xy}] + \sum_{j \in \mathbb{T}} \text{Im}[G_{xj}] |(G|_{\mathbb{T}})^{-1}_{jy}| \\ &+ \sum_{j \in \mathbb{T}} |(G(G|_{\mathbb{T}})^{-1})_{xj}| |\text{Im}[G_{jy}]| + \sum_{j, k \in \mathbb{T}} |G_{xj}| |\text{Im}[(G|_{\mathbb{T}})^{-1}_{jk}]| |G_{ky}|. \end{aligned} \quad (3.36) \quad \{\mathbf{e}:\mathbf{Gimaginar}\}$$

In the following we show the following estimates

$$\sum_{j \in \mathbb{T}} (|G_{xj}| + |(G(G|_{\mathbb{T}})^{-1})_{xj}| + |G_{jy}| + |(G(G|_{\mathbb{T}})^{-1})_{jy}|) \lesssim \ell, \quad \max_{j,k \in \mathbb{T}} \text{Im}[(G|_{\mathbb{T}})^{-1}_{jk}] \lesssim \ell^2 N^{\circ/2} \text{Im}[m]. \quad (3.37) \quad \{\mathbf{e}:\text{sumG}\}$$

Then the second statement in (3.26) follows from plugging (3.24) and (3.37) into (3.36).

We start with the first statement in (3.37). The first statement in (3.37) follows from the estimates (3.33) and (3.34)

$$\sum_{j \in \mathbb{T}} (|G_{xj}| + |(G(G|_{\mathbb{T}})^{-1})_{xj}|) \lesssim \sum_{j \in \mathbb{T}} |G_{xj}| \left(1 + \sum_{i \in \mathbb{T}} |(G|_{\mathbb{T}})^{-1}_{ji}| \right) \lesssim \ell.$$

Next we prove the second statement in (3.37). For the imaginary part of a symmetric matrix, we have the following identity

$$\text{Im}[A^{-1}] = -A^{-1} \text{Im}[A] \overline{A^{-1}}.$$

Thus

$$\text{Im}[(G|_{\mathbb{T}})^{-1}]_{xy} = -((G|_{\mathbb{T}})^{-1} \text{Im}[G|_{\mathbb{T}}] \overline{(G|_{\mathbb{T}})^{-1}})_{xy}. \quad (3.38) \quad \{\mathbf{e}:\text{Im_Ginver}\}$$

The second statement in (3.37) follows from plugging (3.24) and (3.33) into (3.38).

The first statement in (3.27) follows from a Ward identity (2.6), and the second statement in (3.26):

$$\frac{1}{N} \sum_{c \notin \mathbb{T}} |G_{cx}^{(\mathbb{T})}|^2 = \frac{\text{Im}[G_{xx}^{(\mathbb{T})}]}{N\eta} \lesssim \frac{N^{\circ} \text{Im}[m]}{N\eta}.$$

For the second statement in (3.27), thanks to the Schur complement formula (2.5),

$$|G_{cx}^{(b\mathbb{T})}| = \left| G_{cx}^{(\mathbb{T})} - \frac{G_{cb}^{(\mathbb{T})} G_{bx}^{(\mathbb{T})}}{G_{bb}^{(\mathbb{T})}} \right| \lesssim |G_{cx}^{(\mathbb{T})}| + |G_{bx}^{(\mathbb{T})}|, \quad (3.39) \quad \{\mathbf{e}:\text{Gbound}\}$$

where in the second statement we used (3.26) to bound $|G_{cb}^{(\mathbb{T})}| \lesssim 1, |G_{bb}^{(\mathbb{T})}| \gtrsim 1$. By averaging (3.39) we get

$$\frac{1}{Nd} \sum_{b \sim c \notin \mathbb{T}} |G_{cx}^{(b\mathbb{T})}|^2 \lesssim \frac{1}{Nd} \sum_{b \sim c \notin \mathbb{T}} |G_{cx}^{(\mathbb{T})}|^2 + |G_{bx}^{(\mathbb{T})}|^2 \lesssim \frac{N^{\circ} \text{Im}[m]}{N\eta},$$

where the second statement follows from the first statement in (3.27). □

4 Edge universality. \{\mathbf{s}:\text{universali}

We now outline the ideas behind proving edge universality Theorem 1.1 for random d -regular graphs. Edge universality follows the standard three-step scheme [12]. The key deviations arise in steps (i) and (iii):

- (i) In the sparse setting, the optimal concentration of eigenvalue locations is not with respect to the semicircle law as in the Wigner case. The self-consistent equations for the relevant Stieltjes transforms are therefore more intricate. New techniques—local resampling that more effectively exploit the randomness—are introduced to derive these equations and establish concentration; see Section 4.1 for details.
- (iii) For random d -regular graphs with fixed d , standard Wigner-type comparison methods fail. A different mechanism based on loop equations plays a central role; see Section 4.2 for details.

4.1 Three-step strategy.

{s:three_step}

In this section we recall the three-step strategy for proving edge universality for Wigner matrices (see, e.g., [12]) and highlight the key differences that arise in random d -regular graph setting. Let H denote either a Wigner matrix or the normalized adjacency matrix of a random d -regular graph. We write its eigenvalues in decreasing order as $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$.

Step (i): local law and rigidity. The first step is to prove optimal eigenvalue rigidity up to the spectral edge (on the $N^{-2/3}$ scale), yielding precise control of the extreme eigenvalues. This is achieved by deriving a self-consistent equation for the Stieltjes transform $m_N(z)$ (or related spectral observables),

$$m_N(z) := \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z}. \quad (4.1)$$

For Wigner matrices, the self-consistent equation takes the form: with high probability

$$1 + zm_N(z) + m_N^2(z) \approx 0, \quad (4.2) \quad \{\mathbf{e}:\text{self_eq1}\}$$

whose unique solution is the Stieltjes transform of the semicircle law. The self-consistent equation is proven through carefully estimating high moments $\mathbb{E}[|1 + zm_N(z) + m_N^2(z)|^{2p}]$.

For a random d -regular graph $\mathcal{G}$ on N vertices with fixed degree d , obtaining a closed equation for the Stieltjes transform requires a more refined observable. We recall from (3.18) the self-consistent equation

$$Q(z) = \frac{1}{Nd} \sum_{i \sim j} G_{jj}^{(i)}(z), \quad (4.3) \quad \{\mathbf{e}:\text{Qsum0}\}$$

and the functions $Y_\ell(Q(z), z)$ and $X_\ell(Q(z), z)$ from (3.6), which sum the Green's functions at the root of a truncated $(d-1)$ -ary tree and a truncated d -regular tree of depth $\ell \asymp \log_{d-1} N$ with boundary weights $Q(z)$. It turns out that the quantity $Q(z)$ and the Stieltjes transform $m_N(z)$ satisfy two self-consistent equations:

$$Q(z) - Y_\ell(Q(z), z) \approx 0, \quad m_N(z) - X_\ell(Q(z), z) \approx 0. \quad (4.4) \quad \{\mathbf{e}:\text{self_cons}\}$$

One of the main results of [15] is devoted to proving the self-consistent equations (4.4) via a self-consistent scheme utilizing local resampling. This note is devoted to explaining the derivation of these self-consistent equations by showing that their expectations are small.

Step (ii): strong ergodicity of Dyson Brownian motion. We next study the normalized adjacency matrix perturbed by a small GOE matrix

$$H(t) := H + \sqrt{t}Z, \quad (4.5) \quad \{\mathbf{e}:\text{Ht}\}$$

where Z is a GOE matrix for Wigner matrices and Erdős–Rényi graphs; for random d -regular graphs, Z is the GOE restricted to the subspace of matrices with vanishing row and column sums (so that the trivial eigenvector $\mathbf{e} = (1, \dots, 1)^\top$ is preserved).

Assume the eigenvalues of H satisfy optimal edge rigidity with respect to a limiting density ϱ exhibiting square-root edge behavior, which is the consequence of Step (i). Then, by [1, 20], after a time $t \geq N^{-1/3+\mathfrak{c}}$ (for arbitrarily small $\mathfrak{c} > 0$) the edge statistics of $H(t)$ are universal: the extreme eigenvalues converge to the Airy_1 point process (the edge scaling limit of GOE). In particular, for Wigner matrices ϱ is the semicircle law (1.5); for random d -regular graphs ϱ is the Kesten–McKay law ϱ_d from (1.3).

The asymptotic empirical eigenvalue distribution of the matrix $H(t) = H + \sqrt{t}Z$ can be described by the free additive convolution, from free probability theory [5]. We denote the semicircle distribution of variance t as $t^{-1/2}\rho_{\text{sc}}(t^{-1/2}x)$. Given a probability measure ϱ on $\mathbb{R}$, we denote its free convolution with a semicircle distribution of variance t by $\varrho_t := \varrho \boxplus t^{-1/2}\rho_{\text{sc}}(t^{-1/2}x)$. Write the Stieltjes transforms of ϱ and ϱ_t as

$$m(z) = m(z; 0) = \int (x - z)^{-1} \varrho(x) dx, \quad m(z; t) = \int (x - z)^{-1} \varrho_t(x) dx, \quad z \in \mathbb{C} \setminus \mathbb{R}.$$

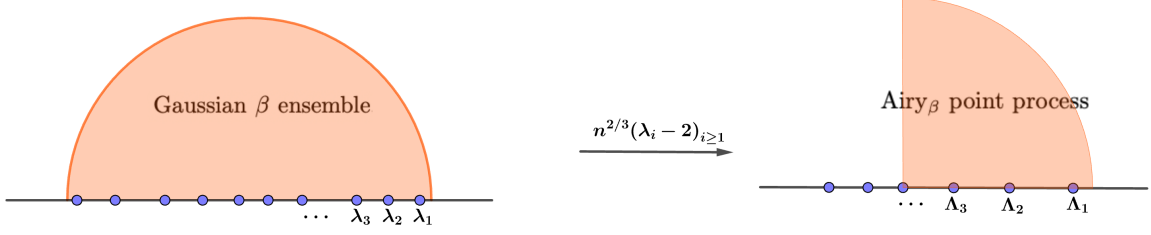


Figure 2: Extreme eigenvalues of Gaussian β -ensemble converge to the Airy $_{\beta}$ point process.

{f:airy}

The Stieltjes transform $m(z; t)$ solves the complex Burgers equation

$$\partial_t m(z, t) = \partial_z m(z, t) m(z, t) = \frac{1}{2} \partial_z (m(z, t)^2). \quad (4.6) \quad \{\mathbf{e:eq2}\}$$

We denote the right edge of the free convolution density ρ_t as E_t , then its evolution satisfies the following equation

$$\partial_t E_t = m(E_t, t). \quad (4.7) \quad \{\mathbf{e:Etshift}\}$$

Step (iii): Green function comparison. This final step removes the Gaussian component in (4.5) and, via a comparison argument, transfers edge universality from the Gaussian-divisible model to the original ensemble. In particular, it shows that the k -point correlation function at the edge is invariant along the flow (4.5), with t interpreted as time. This is achieved by the following comparison theorem for multipoint correlation functions at the edge (equivalently, by comparing products of Stieltjes transforms on the microscopic scale). For details on how this theorem yields edge universality, see [12, Section 17].

{e:Etshift}

{p:small_chan}

Proposition 4.1. Fix a small $\mathfrak{c} > 0$. We introduce the microscopic window:

$$\mathbf{M} := \{w \in \mathbb{C} : N^{-2/3-\mathfrak{c}} \leq |\operatorname{Im}[w]| \leq N^{-2/3+\mathfrak{c}}, -N^{-2/3+\mathfrak{c}} \leq \operatorname{Re}[w] \leq N^{-2/3+\mathfrak{c}}\}. \quad (4.8) \quad \{\mathbf{e:micro_win}\}$$

Then for any $p \geq 1$, and $w_1, w_2, \dots, w_p \in \mathbf{M}$, $z_j(t) = E_t + w_j$ for $1 \leq j \leq p$, the following holds

$$\mathbb{E} \left[\prod_{j=1}^p N^{1/3} (m_N(z_j(t); t) - m(z_j(t); t)) \right] \Bigg|_{t=0}^{t=N^{-1/3+\mathfrak{c}}} = O(N^{-\mathfrak{c}}), \quad (4.9) \quad \{\mathbf{e:product_s}\}$$

provided N is sufficiently large.

For Wigner matrices and random graphs with polynomially growing degree, Proposition 4.1 follows by integrating out the GOE perturbation Z via Gaussian integration by parts, expanding the randomness of H via cumulant expansion, and showing that the leading contributions cancel. For random d -regular graphs with fixed degree d , this argument breaks down. In the proof of Theorem 1.1, a key observation is that the time derivative of the multi-point correlation functions of the Stieltjes transform of $H(t)$ is governed by microscopic loop equations. In the next section, we state these loop equations and explain how they yield the Green function comparison for fixed-degree d -regular graphs.

4.2 Loop equations and Green's function comparison.

{s:loop1}

Loop (or Dyson-Schwinger) equations were first used in the theoretical physics literature (e.g., in the work of Migdal [22]) and were later introduced to the mathematical community by Johansson [18] to derive macroscopic central limit theorems for general β -ensembles of eigenvalues of random matrices, see also [6, 7, 25] and the book [13]. For the Gaussian β -ensemble, the loop (or Dyson-Schwinger) equation describes

a recursive structure satisfied by the Stieltjes transform (correlation functions) of the eigenvalues. The first loop equation is given by : for $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\mathbb{E} \left[m_N(z)^2 + z m_N(z) + 1 + \frac{2 - \beta}{\beta} \frac{\partial_z m_N(z)}{N} \right] = 0, \quad m_N(z) := \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z}. \quad (4.10) \quad \{\mathbf{e:loop0}\}$$

More generally, for any $p \geq 1$, the loop equations at rank p are given by : for $z, z_1, z_2, \dots, z_{p-1} \in \mathbb{C} \setminus \mathbb{R}$

$$\begin{aligned} & \mathbb{E} \left[\left(m_N(z)^2 + z m_N(z) + 1 + \frac{2 - \beta}{\beta} \frac{\partial_z m_N(z)}{N} \right) \prod_{j=1}^{p-1} m_N(z_j) \right] \\ & + \frac{2}{\beta N^2} \sum_{j=1}^{p-1} \mathbb{E} \left[\partial_{z_j} \frac{m_N(z) - m_N(z_j)}{z - z_j} \prod_{i \neq j} m_N(z_i) \right] = 0. \end{aligned} \quad (4.11) \quad \{\mathbf{e:loop1}\}$$

These loop equations (4.10) and (4.11) arise from the invariance of the eigenvalue integral under infinitesimal reparametrizations of the variables; they can be derived either by integration by parts or, equivalently, by an infinitesimal change of variables.

More generally, one can derive the loop equations for β -ensembles with general potential V

$$P(\lambda_1, \lambda_2, \dots, \lambda_N) = \frac{1}{Z_N^{\beta, V}} \prod_{i < j} |\lambda_i - \lambda_j|^\beta \prod_{i=1}^N e^{-\beta N V(\lambda_i)}.$$

Loop (or Dyson-Schwinger) equations play a crucial role in proving eigenvalue rigidity and universality for β -ensembles with general potential [4, 8–10, 26].

In general, loop equations are model-dependent. For β -ensembles, they depend on the potential V . However, when zooming in near the spectral edge, the microscopic version of loop equations becomes universal. To introduce *microscopic version of loop equations*, we recall that the extreme eigenvalues of GOE converge to the Airy_1 point process $\Lambda_1 \geq \Lambda_2 \geq \Lambda_3 \geq \dots$; see Figure 2. We can introduce the following normalized Stieltjes transform of the Airy_1 point process:

$$S(w) = \sum_{i \geq 1} \frac{1}{\Lambda_i - w} - \frac{1}{\mathfrak{a}_i} - \frac{\text{Ai}'(0)}{\text{Ai}(0)}, \quad w \in \mathbb{C} \setminus \mathbb{R},$$

where $\mathfrak{a}_1 > \mathfrak{a}_2 > \mathfrak{a}_3 > \dots$ are zeros of the Airy function $\text{Ai}(w)$. Under the above normalization, $S(w)$ has square root behavior: $S(w) \sim \sqrt{w}$ as $w \rightarrow \infty$. $S(w)$ can be viewed as a random meromorphic function, with poles given by the Airy_1 point process.

Formally, taking the soft-edge scaling limit of the loop equations (4.10) and (4.11), we obtain a hierarchy of equations for the Stieltjes transform $S(w)$ of the Airy_1 point process. The rank- p loop equations are given by: for any $w, w_1, \dots, w_{p-1} \in \mathbb{C} \setminus \mathbb{R}$,

$$\begin{aligned} & \mathbb{E} \left[\left((S(w) - \sqrt{w})^2 + 2\sqrt{w}(S(w) - \sqrt{w}) + \partial_w S(w) \right) \prod_{i=1}^{p-1} S(w_i) \right] \\ & + 2 \sum_{1 \leq i \leq p-1} \mathbb{E} \left[\partial_{w_i} \left(\frac{S(w) - S(w_i)}{w - w_i} \right) \prod_{j: j \neq i} S(w_j) \right] = 0. \end{aligned} \quad (4.12) \quad \{\mathbf{e:Airy}\}$$

These loop equations (4.12) provide necessary conditions for the edge statistics to converge to the Airy point process.

A key intermediate step in [15] toward the proof of Theorem 1.1 is the following microscopic loop equation for the normalized adjacency matrix of a random d -regular graph perturbed by a small GOE. Its proof occupies the main part of [15] and is analogous to the derivation of the self-consistent equations (4.4). To make the error terms sufficiently small, we need to iterate the local resampling procedure many times and carefully track the accumulated errors.

{p:loop}

Proposition 4.2. Fix a small $\mathfrak{c} > 0$. Let $\Delta_t(z) := m_N(z; t) - m(z; t)$, and recall the microscopic window $\mathbf{M}$ from (4.8). Fix $p \geq 1$ and $w, w_1, \dots, w_{p-1} \in \mathbf{M}$, and set $z(t) := E_t + w$ and $z_j(t) := E_t + w_j$ for $1 \leq j \leq p-1$. Then for N sufficiently large,

$$\begin{aligned} & \mathbb{E} \left[\left(\Delta_t(z(t))^2 + 2\mathcal{A}\sqrt{z(t) - E_t} \Delta_t(z(t)) + \frac{\partial_z m_N(z(t); t)}{N} \right) \prod_{i=1}^{p-1} \Delta_t(z_i(t)) \right] \\ & + \frac{2}{N^2} \sum_{1 \leq i \leq p-1} \mathbb{E} \left[\partial_{z_i(t)} \left(\frac{m_N(z(t); t) - m_N(z_i(t); t)}{z(t) - z_i(t)} \right) \prod_{j: 1 \leq j \neq i \leq p-1} \Delta_t(z_j(t)) \right] = O \left(N^{-(p+1)/3-10\mathfrak{c}} \right). \end{aligned} \quad (4.13) \quad \{\mathfrak{e}:micro_loop\}$$

The scaling limit $N^{1/3} \Delta_t(E_t + w/(\mathcal{A}N)^{2/3})/\mathcal{A}^{2/3}$ converges to $S(w) - \sqrt{w}$ in the microscopic version of loop equations (4.12). The micro-loop on the right-hand side of (4.13) is precisely calibrated so that, after scaling each factor of Δ_t by $N^{1/3}$, the total error still vanishes as $N \rightarrow \infty$.

In what follows, we show that the microscopic loop equations (4.13) imply the Green's function comparison in Proposition 4.1, thereby completing the final step of the three-step strategy. For clarity, we prove (4.9) only in the case $p = 1$, writing $z_1(t) = z(t) = E_t + w$ with $w \in \mathbf{M}$. We refer to [15, Section 3.3] for the general case. In this case, (4.9) follows from integrating the following statement from 0 to $t = N^{-1/3+\mathfrak{c}}$

$$|\partial_t \mathbb{E}[m_N(z(t); t) - m(z(t); t)]| \leq N^{1/3-2\mathfrak{c}}. \quad (4.14) \quad \{\mathfrak{e}:small_change\}$$

The first loop equation corresponds to $p = 1$ in (4.13) {e:small_change}

$$\mathbb{E} \left[(\Delta_t(z(t))^2 + 2\mathcal{A}\sqrt{z(t) - E_t} \Delta_t(z(t)) + \frac{\partial_z m_N(z(t); t)}{N} \right] = O \left(N^{-1/3-10\mathfrak{c}} \right), \quad (4.15) \quad \{\mathfrak{e}:firsteq\}$$

where $z(t) = E_t + w$ with $w \in \mathbf{M}$. Next we show that (4.15) implies (4.14). By Gaussian integration by parts,

$$\partial_t \mathbb{E}[m_N(z; t)] = \frac{1}{N} \partial_t \mathbb{E}[\text{Tr}(H + \sqrt{t}Z - z)^{-1}] = \frac{1}{2} \mathbb{E} \left[\partial_z \left(m_N^2(z; t) + \frac{\partial_z m_N(z; t)}{N} \right) \right]. \quad (4.16) \quad \{\mathfrak{e}:eq1\}$$

Subtracting the complex Burgers equation (4.6) from (4.16) gives

$$\partial_t \mathbb{E}[m_N(z; t) - m(z; t)] = \frac{1}{2} \mathbb{E} \left[\partial_z \left(m_N^2(z; t) - m^2(z; t) + \frac{\partial_z m_N(z; t)}{N} \right) \right]. \quad (4.17) \quad \{\mathfrak{e}:DBM_mt\}$$

Plugging $z(t) = E_t + w$ with $w \in \mathbf{M}$ into (4.17), we obtain

$$\begin{aligned} \partial_t \mathbb{E}[m_N(z(t); t) - m(z(t); t)] &= \frac{1}{2} \mathbb{E}[\partial_z F_t(z(t))], \\ F_t(z) &:= \Delta_t(z)^2 + 2(m(z; t) - m(E_t; t))\Delta_t(z) + \frac{\partial_z m_N(z; t)}{N}. \end{aligned} \quad (4.18) \quad \{\mathfrak{e}:DBM_mt2\}$$

For z near the spectral edge, with $z - E_t \in \mathbf{M}$, and using the square-root behavior of ϱ_t , we have $m(z; t) - m(E_t; t) \approx \mathcal{A}\sqrt{z - E_t}$. Thus, neglecting the replacement error, the first loop equation (4.15) implies

$$\mathbb{E}[F_t(z)] = O \left(N^{-1/3-10\mathfrak{c}} \right). \quad (4.19) \quad \{\mathfrak{e}:F_t\}$$

Since F_t is analytic on $\mathbb{C} \setminus \mathbb{R}$, Cauchy's integral formula on a circle ω centered at $z(t)$ with radius $\text{Im } z(t)/2 \geq N^{-2/3-\mathfrak{c}}/2$ yields

$$\mathbb{E}[\partial_z F_t(z(t))] = \frac{1}{2\pi i} \oint_{\omega} \frac{\mathbb{E}[F_t(w)]}{(w - z(t))^2} dw = O \left(N^{1/3-8\mathfrak{c}} \right). \quad (4.20) \quad \{\mathfrak{e}:F_t2\}$$

Combining (4.18) with (4.20) proves (4.14), and hence the Green function comparison Proposition 4.1.

4.3 Self-consistent equation and the first loop equation

To illustrate the main ideas, in these notes we present the proof of Theorem 4.5, which yields the self-consistent equations and the first microscopic version of the loop equations at the edge. We refer to [15] for the complete proof of edge universality.

We recall the functions Y_ℓ, X_ℓ and the quantity $Q(z)$ from (3.6) and (3.18). For simplicity of notation, we write

$$Y(z) = Y_\ell(Q(z), z), \quad X(z) = X_\ell(Q(z), z). \quad (4.21) \quad \{\mathbf{e}:\mathbf{defYt}\}$$

We also recall the sets Ω and $\bar{\Omega}$ of d -regular graphs from Definition 3.1 and Theorem 3.6.

Next, we introduce some further error terms. We emphasize that they depend on the graph $\mathcal{G}$ and are therefore random quantities. \{\mathbf{def:phidef}\}

Definition 4.3. For any $z \in \mathbb{C}^+$ in the upper half-plane with $\eta = \text{Im } z$, we introduce the control parameter

$$\Phi(z) := \frac{\text{Im}[m_N(z)]}{N\eta} + \frac{1}{N^{1-2\epsilon}}, \quad \Psi(z) := \mathbf{1}(\mathcal{G} \in \Omega) \left(\Phi(z) + \frac{|Q(z) - Y(z)|}{N^{b/8}} \right). \quad (4.22) \quad \{\mathbf{e}:\mathbf{defPhi}\}$$

Remark 4.4. The quantity $\Phi(z)$ is a standard control parameter in random matrix theory. We will control the expectation of $Q(z) - Y(z)$ via $\Psi(z)$, see (4.23). ~~At the edge, $\Psi(z)$ itself contains $|Q(z) - Y(z)|$ (but with an extra factor $N^{-b/8}$), we can (ignoring the absolute value) move this term to the left-hand side and conclude that the expectation of $Q(z) - Y(z)$ is controlled by $\Phi(z)$.~~

The following result states that the expectation of the self-consistent equations is negligible, and it gives the first loop equation corresponding to the case $t = 0, p = 1$ of Proposition 4.2. The proof of Theorem 4.5 will be given in the remainder of this article. \{\mathbf{t:recursion}\}

Theorem 4.5. Fix a spectral parameter $z \in \mathbf{D}$, and recall the control parameter $\Psi(z)$ from (4.22). Then

$$\mathbb{E} [\mathbf{1}(\mathcal{G} \in \Omega) (Q(z) - Y(z))] \lesssim (d-1)^{2\ell} \mathbb{E} [\Psi(z)], \quad (4.23) \quad \{\mathbf{e}:\mathbf{QY}\}$$

$$\mathbb{E} [\mathbf{1}(\mathcal{G} \in \Omega) (m_N(z) - X(z))] \lesssim (d-1)^{2\ell} \mathbb{E} [\Psi(z)]. \quad (4.24) \quad \{\mathbf{e}:\mathbf{mz}\}$$

Moreover, if we further assume that $|z - 2| \leq N^{-\mathfrak{g}}$, then we have a refined estimate for $Q - Y$:

$$\mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \left(\frac{\mathcal{A}^2}{\ell+1} (Q(z) - Y(z)) + \frac{\partial_z m_N(z)}{N} \right) \right] = \mathcal{O} \left(\frac{N^{\circ} \mathbb{E} [\Psi(z)]}{(d-1)^{\ell/2}} \right), \quad (4.25) \quad \{\mathbf{e}:\mathbf{Qrefined}\}$$

where the constant $\mathcal{A} = d(d-1)/(d-2)^2$ is as in (1.4). ~~If $|z - 2| \leq N^{-\mathfrak{g}}$, an analogous statement holds after multiplying the first term in (4.25) by -1 .~~

Close to the right spectral edge, i.e. for $|z - 2| \lesssim N^{-2/3+o(1)}$ with $\text{Im } z \gtrsim N^{-2/3+o(1)}$, we have

$$\begin{aligned} m_{\text{sc}}(z) &= -1 + \mathcal{O}(\sqrt{|z-2|}), \quad 1 - m_{\text{sc}}^2(z) = 2\sqrt{z-2} + \mathcal{O}(|z-2|), \\ m_d(z) &= -\frac{d-1}{d-2} + \mathcal{O}(\sqrt{|z-2|}), \end{aligned} \quad (4.26) \quad \{\mathbf{e}:\mathbf{mscmd0}\}$$

and the true sizes of the error terms are

$$\text{Im}[m_N(z)] \lesssim \frac{N^{\circ(1)}}{N^{1/3}}, \quad \Phi(z), \Psi(z) \lesssim \frac{N^{\circ(1)}}{N^{2/3}}. \quad (4.27) \quad \{\mathbf{e}:\mathbf{edge}\}$$

The optimal edge rigidity of eigenvalues in [15] follows from establishing analogous high-moment estimates to (4.23) and (4.24), which essentially lead to

$$|Q(z) - Y(z)| \lesssim \frac{N^{\circ(1)}}{N^{2/3}}, \quad |m_N(z) - X(z)| \lesssim \frac{N^{\circ(1)}}{N^{1/2}}. \quad (4.28) \quad \{\mathbf{e}:\mathbf{QYmX}\}$$

We remark that, for technical reasons, the estimate for $|m_N(z) - X(z)|$ is weaker. Nevertheless, these estimates are sufficient to derive optimal bounds for $Q(z) - m_{\text{sc}}(z)$ and $m_N(z) - m_d(z)$, as explained below.

The concentration of $Q(z)$ and $m_N(z)$ can be derived from (4.28). We recall the expansion of $Y(z) = Y_\ell(Q(z), z)$ from (3.7) (by taking $\Delta = Q(z)$):

$$\begin{aligned}
Q - Y &= (Q - m_{\text{sc}}(z)) - (Y - m_{\text{sc}}(z)) = (1 - m_{\text{sc}}^{2(\ell+1)}(z))(Q - m_{\text{sc}}(z)) \\
&\quad - m_{\text{sc}}^{2\ell+2}(z)m_d(z) \left(\frac{1 - m_{\text{sc}}^{2\ell+2}(z)}{d-1} + \frac{d-2}{d-1} \frac{1 - m_{\text{sc}}^{2\ell+2}(z)}{1 - m_{\text{sc}}^2(z)} \right) (Q - m_{\text{sc}}(z))^2 + O(\ell^5 |Q - m_{\text{sc}}(z)|^3) \\
&= (1 - m_{\text{sc}}^2(z))(\ell+1)(Q - m_{\text{sc}}(z)) + (\ell+1)(Q - m_{\text{sc}}(z))^2 \\
&\quad + O\left(\ell(|z-2||Q - m_{\text{sc}}(z)| + \sqrt{|z-2|}|Q - m_{\text{sc}}(z)|^2) + \ell^5 |Q - m_{\text{sc}}(z)|^3\right) \\
&= (\ell+1)\left(2\sqrt{z-2}(Q - m_{\text{sc}}(z)) + (Q - m_{\text{sc}}(z))^2\right) \\
&\quad + O\left(\ell(|z-2||Q - m_{\text{sc}}(z)| + \sqrt{|z-2|}|Q - m_{\text{sc}}(z)|^2) + \ell^5 |Q - m_{\text{sc}}(z)|^3\right),
\end{aligned} \tag{4.29}$$

where we used (4.26) to replace $m_{\text{sc}}(z), m_d(z), 1 - m_{\text{sc}}^2(z)$ by $-1, -(d-1)/(d-2), 2\sqrt{z-2}$, respectively. Given that $Q - Y$ is small as in (4.28), we can analyze (4.29) as a quadratic equation in $Q - m_{\text{sc}}(z)$ with a small error. Such quadratic equations are ubiquitous in random matrix theory; for example, for Wigner matrices the standard argument shows that for $|z-2| \lesssim N^{-2/3+o(1)}$, with high probability we have

$$|Q - m_{\text{sc}}(z)| \lesssim \frac{N^{o(1)}}{N^{1/3}}. \tag{4.30}$$

Using the expansion of $X(z) = X_\ell(Q(z), z)$ from (3.8) (with $\Delta = Q(z)$), we have

$$\begin{aligned}
X - m_d(z) &= \frac{d}{d-1} m_d^2(z) m_{\text{sc}}^{2\ell}(z) (Q - m_{\text{sc}}(z)) + O(|Q - m_{\text{sc}}(z)|^2) \\
&= \mathcal{A}(Q - m_{\text{sc}}(z)) + O(\ell\sqrt{|z-2|}|Q - m_{\text{sc}}(z)| + |Q - m_{\text{sc}}(z)|^2),
\end{aligned}$$

where we again used (4.26) to replace $m_{\text{sc}}(z), m_d(z)$ by $-1, -(d-1)/(d-2)$. It follows, by rearranging, that

$$\begin{aligned}
m_N(z) - m_d(z) &= m_N(z) - X + X - m_d(z) \\
&= \mathcal{A}(Q - m_{\text{sc}}(z)) + O(|m_N(z) - X| + \ell\sqrt{|z-2|}|Q - m_{\text{sc}}(z)| + |Q - m_{\text{sc}}(z)|^2).
\end{aligned} \tag{4.31}$$

Combining (4.28), (4.30) and (4.31), we conclude that for $|z-2| \lesssim N^{-2/3+o(1)}$, with high probability

$$m_N(z) - m_d(z) = \mathcal{A}(Q - m_{\text{sc}}(z)) + O\left(\frac{N^{o(1)}}{N^{1/2}}\right) = O\left(\frac{N^{o(1)}}{N^{1/3}}\right). \tag{4.32}$$

We remark that (4.25) is slightly different from the loop equation (4.13) (with $t = 0$ and $p = 1$):

$$\mathbb{E}\left[(m_N(z) - m_d(z))^2 + 2\mathcal{A}\sqrt{z-2}(m_N(z) - m_d(z)) + \frac{\partial_z m_N(z)}{N}\right] = O\left(N^{-2/3-10\epsilon}\right). \tag{4.33}$$

By plugging (4.31) into (4.33), we conclude that for $|z-2| \lesssim N^{-2/3+o(1)}$, with high probability

$$\frac{\mathcal{A}^2(Q - Y)}{\ell+1} = (m_N(z) - m_d(z))^2 + 2\mathcal{A}\sqrt{z-2}(m_N(z) - m_d(z)) + O\left(\frac{N^{o(1)}}{N^{5/6}}\right). \tag{4.34}$$

The claim (4.33) then follows from (4.25), and the error term comes from

$$\frac{N^o \mathbb{E}[\Psi(z)]}{(d-1)^{\ell/2}} \lesssim \frac{N^{o+o(1)}}{(d-1)^{\ell/2} N^{2/3}} \lesssim N^{-2/3-10\epsilon}, \tag{4.35}$$

where we used (4.27).

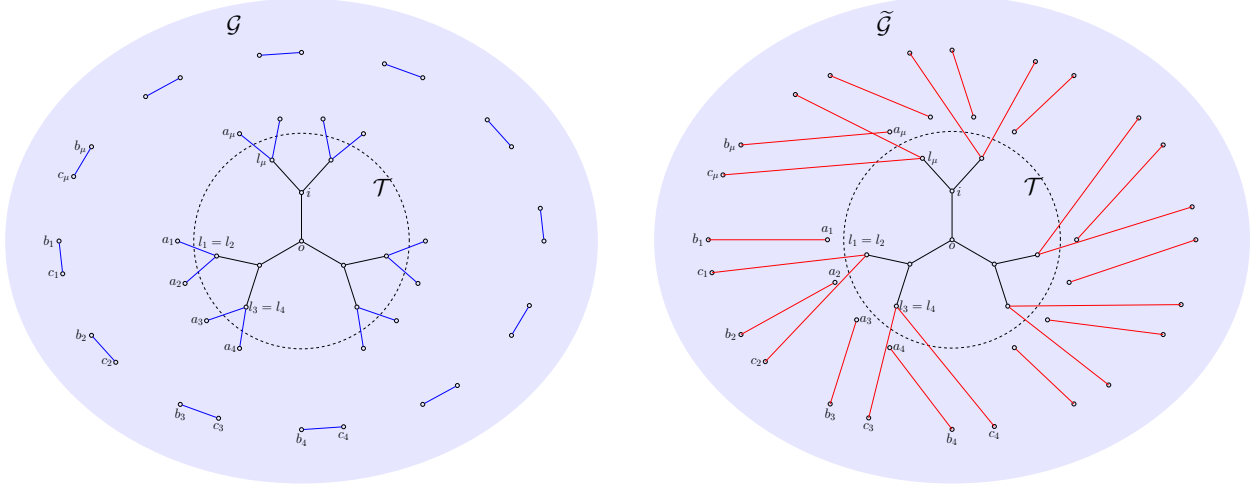


Figure 3: An example of the local resampling performed on the graph is as follows. We replace the blue edges, located on the boundary of the radius- ℓ neighborhood of a vertex o , with randomly chosen edges from the graph. Together, these edges constitute the resampling data, denoted by $\mathbf{S}$. This operation creates new red edges and establishes a new boundary.

{fig:switching}

5 Local resampling

{s:local_resa}

In this section, we recall the local resampling and its properties. This gives us the framework to talk about resampling from the random regular graph distribution as a way to get an improvement in our estimates of the Green's function.

For any graph $\mathcal{G}$, we denote the set of unoriented edges by $E(\mathcal{G})$, and the set of oriented edges by $\vec{E}(\mathcal{G}) := \{(u, v), (v, u) : \{u, v\} \in E(\mathcal{G})\}$. For a subset $\vec{S} \subset \vec{E}(\mathcal{G})$, we denote by S the set of corresponding non-oriented edges. For a subset $S \subset E(\mathcal{G})$ of edges we denote by $[S] \subset \llbracket N \rrbracket$ the set of vertices incident to any edge in S . Moreover, for a subset $\mathbb{V} \subset \llbracket N \rrbracket$ of vertices, we define $E(\mathcal{G})|_{\mathbb{V}}$ to be the subgraph of $\mathcal{G}$ induced by $\mathbb{V}$.

Definition 5.1. A (simple) switching is encoded by two oriented edges $\vec{S} = \{(v_1, v_2), (v_3, v_4)\} \subset \vec{E}$. We assume that the two edges are disjoint, i.e. that $|\{v_1, v_2, v_3, v_4\}| = 4$. Then the switching consists of replacing the edges $\{v_1, v_2\}, \{v_3, v_4\}$ with the edges $\{v_1, v_4\}, \{v_2, v_3\}$. We denote the graph after the switching $\vec{S}$ by $T_{\vec{S}}(\mathcal{G})$, and the new edges $\vec{S}' = \{(v_1, v_4), (v_2, v_3)\}$ by $T(\vec{S}) = \vec{S}'$.

The local resampling involves a fixed center vertex, which we now assume to be vertex o , and a radius ℓ . Given a d -regular graph $\mathcal{G}$, we write $\mathcal{T} := \mathcal{B}_\ell(o, \mathcal{G})$ to denote the radius- ℓ neighborhood of o (which may not necessarily be a tree) and write $\mathbb{T}$ for its vertex set. The edge boundary $\partial_E \mathcal{T}$ of $\mathcal{T}$ consists of the edges in $\mathcal{G}$ with one vertex in $\mathbb{T}$ and the other vertex in $\llbracket N \rrbracket \setminus \mathbb{T}$. We enumerate the edges of $\partial_E \mathcal{T}$ as $\partial_E \mathcal{T} = \{e_1, e_2, \dots, e_\mu\}$, where $e_\alpha = \{l_\alpha, a_\alpha\}$ with $l_\alpha \in \mathbb{T}$ and $a_\alpha \in \llbracket N \rrbracket \setminus \mathbb{T}$. We orient the edges e_α by defining $\vec{e}_\alpha = (l_\alpha, a_\alpha)$. We notice that μ and the edges $e_1, e_2, \dots, e_\mu$ depend on $\mathcal{G}$. The edges e_α are distinct, but the vertices a_α are not necessarily distinct and neither are the vertices l_α . Our local resampling switches the edge boundary of $\mathcal{T}$ with randomly chosen edges in $\mathcal{G}^{(\mathbb{T})}$ if the switching is admissible (see below), and leaves them in place otherwise. To perform our local resampling, see Figure 3, we choose $(b_1, c_1), \dots, (b_\mu, c_\mu)$ to be independent, uniformly chosen oriented edges from the graph $\mathcal{G}^{(\mathbb{T})}$, i.e., the oriented edges of $\mathcal{G}$ that are not incident to $\mathbb{T}$, and define

$$\vec{S}_\alpha = \{\vec{e}_\alpha, (b_\alpha, c_\alpha)\}, \quad \mathbf{S} = (\vec{S}_1, \vec{S}_2, \dots, \vec{S}_\mu). \quad (5.1) \quad \{\mathbf{e: defSa}\}$$

The sets $\mathbf{S}$ will be called the *resampling data* for $\mathcal{G}$. We remark that repetitions are allowed in the data $(b_1, c_1), (b_2, c_2), \dots, (b_\mu, c_\mu)$. We define an indicator that will be crucial to the definition of the switch.

Definition 5.2. For $\alpha \in \llbracket \mu \rrbracket$, we define the indicator functions $I_\alpha \equiv I_\alpha(\mathcal{G}, \mathbf{S}) = 1$

1. the subgraph $\mathcal{B}_{\Re/4}(\{a_\alpha, b_\alpha, c_\alpha\}, \mathcal{G}^{(\mathbb{T})})$ after adding the edge $\{a_\alpha, b_\alpha\}$ is a tree;
2. and $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(\{a_\alpha, b_\alpha, c_\alpha\}, \{a_\beta, b_\beta, c_\beta\}) > \Re/4$ for all $\beta \in \llbracket \mu \rrbracket \setminus \{\alpha\}$.

The indicator function I_α imposes two conditions. The first one is a “tree” condition, which ensures that a_α and $\{b_\alpha, c_\alpha\}$ are far away from each other, and their neighborhoods are trees. The second one imposes an “isolation” condition, which ensures that we only perform simple switching when the switching pair is far away from other switching pairs. In this way, we do not need to keep track of the interaction between different simple switchings.

We define the *admissible set*

$$\mathbb{W}_{\mathbf{S}} := \{\alpha \in \llbracket \mu \rrbracket : I_\alpha(\mathcal{G}, \mathbf{S})\}. \quad (5.2) \quad \{\{\mathbf{Wdef}\}\}$$

We say that the index $\alpha \in \llbracket \mu \rrbracket$ is *switchable* if $\alpha \in \mathbb{W}_{\mathbf{S}}$. We denote the set $\mathbb{W}_{\mathbf{S}} = \{b_\alpha : \alpha \in \mathbb{W}_{\mathbf{S}}\}$. Let $\nu := |\mathbb{W}_{\mathbf{S}}|$ be the number of admissible switchings and $\alpha_1, \alpha_2, \dots, \alpha_\nu$ be an arbitrary enumeration of $\mathbb{W}_{\mathbf{S}}$. Then we define the switched graph by

$$T_{\mathbf{S}}(\mathcal{G}) := \left(T_{\vec{S}_{\alpha_1}} \circ \dots \circ T_{\vec{S}_{\alpha_\nu}} \right) (\mathcal{G}), \quad (5.3) \quad \{\{\mathbf{e:Tdef1}\}\}$$

and the resampling data by

$$T(\mathbf{S}) := (T_1(\vec{S}_1), \dots, T_\mu(\vec{S}_\mu)), \quad T_\alpha(\vec{S}_\alpha) := \begin{cases} T(\vec{S}_\alpha) & (\alpha \in \mathbb{W}_{\mathbf{S}}), \\ \vec{S}_\alpha & (\alpha \notin \mathbb{W}_{\mathbf{S}}). \end{cases} \quad (5.4) \quad \{\{\mathbf{e:Tdef2}\}\}$$

To make the structure more clear, we introduce an enlarged probability space. Equivalent to the definition above, the sets $\vec{S}_\alpha$ as defined in (5.1) are uniformly distributed over

$$\mathcal{S}_\alpha(\mathcal{G}) = \{\vec{S} \subset \vec{E} : \vec{S} = \{\vec{e}_\alpha, \vec{e}\}, \vec{e} \text{ is not incident to } \mathcal{T}\},$$

i.e., the set of pairs of oriented edges in $\vec{E}$ containing $\vec{e}_\alpha$ and another oriented edge in $\mathcal{G}^{(\mathbb{T})}$. Therefore $\mathbf{S} = (\vec{S}_1, \vec{S}_2, \dots, \vec{S}_\mu)$ is uniformly distributed over the set $\mathcal{S}(\mathcal{G}) = \mathcal{S}_1(\mathcal{G}) \times \dots \times \mathcal{S}_\mu(\mathcal{G})$.

We introduce the following notation on the probability and expectation with respect to the randomness of the $\mathbf{S} \in \mathcal{S}(\mathcal{G})$. \{\mathbf{def:PS}\}

Definition 5.3. Given any d -regular graph $\mathcal{G}$, we denote $\mathbb{P}_{\mathbf{S}}(\cdot)$ the uniform probability measure on $\mathcal{S}(\mathcal{G})$; and $\mathbb{E}_{\mathbf{S}}[\cdot]$ the expectation over the choice of $\mathbf{S}$ according to $\mathbb{P}_{\mathbf{S}}$.

The following claim from [17, Lemma 7.3] states that this switch is invariant under the random regular graph distribution. \{\mathbf{lem:exchange}\}

Lemma 5.4 ([17, Lemma 7.3]). Fix $d \geq 3$. We recall the operator $T_{\mathbf{S}}$ from (5.3). Let $\mathcal{G}$ be a random d -regular graph and $\mathbf{S}$ uniformly distributed over $\mathcal{S}(\mathcal{G})$, then the graph pair $(\mathcal{G}, T_{\mathbf{S}}(\mathcal{G}))$ forms an exchangeable pair:

$$(\mathcal{G}, T_{\mathbf{S}}(\mathcal{G})) \stackrel{\text{law}}{=} (T_{\mathbf{S}}(\mathcal{G}), \mathcal{G}).$$

5.1 Green’s function of resampled graph

Fix an edge $\{i, o\} \subset \mathcal{G}$, we recall the resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in \llbracket \mu \rrbracket}$ around o from Section 5, denote $\tilde{\mathcal{G}} = T_{\mathbf{S}}\mathcal{G}$. In the remainder of the paper, we denote by $\tilde{H}$ the normalized adjacency matrix of $\tilde{\mathcal{G}}$. Then

$$\tilde{H} - H = - \sum_{\alpha \in \llbracket \mu \rrbracket} \xi_\alpha, \quad \xi_\alpha := \frac{1}{\sqrt{d-1}} (\Delta_{l_\alpha a_\alpha} + \Delta_{b_\alpha c_\alpha} - \Delta_{l_\alpha c_\alpha} - \Delta_{a_\alpha b_\alpha}). \quad (5.5) \quad \{\{\mathbf{e:H-H}\}\}$$

Its Green's function and the Stieltjes transform of its empirical eigenvalue distribution are denoted as follows:

$$\tilde{G}(z) = (\tilde{H} - z)^{-1}, \quad \tilde{m}_N(z) = \frac{1}{N} \text{Tr} \tilde{G}(z). \quad (5.6) \quad \{\mathbf{e:tGtm}\}$$

In the rest of this section we collect some basic estimates of the Green's functions of $G(z), \tilde{G}(z)$. Their proofs follow from analyzing Green's functions using the Schur complement formula (2.4). $\{\mathbf{e:tGtm}\}$ $\{\mathbf{c:expectatio}$

Lemma 5.5. *Let $z \in \mathbb{C}^+$ satisfy $|z| \leq 1/\mathfrak{g}$, and set $\eta := \text{Im } z \geq N^{-1+\mathfrak{g}}$. Fix a d -regular graph $\mathcal{G} \in \Omega$ (recall Definition 3.7) and vertices $\{i, o\}$ in $\mathcal{G}$, and let $\mathcal{T} = \mathcal{B}_\ell(o, \mathcal{G})$ with vertex set $\mathbb{T}$. We denote the resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$ around o (recall from Section 5), and $\tilde{\mathcal{G}} = T_{\mathbf{S}}\mathcal{G}$. Moreover, we also assume that $\tilde{\mathcal{G}} \in \Omega$.*

Then, for any $\mathbb{X}$ of the form $\mathbb{X} \in \{\emptyset, \mathbb{T}\}$ and any $x, y \notin \mathbb{X}$, we have

$$|\tilde{G}_{xy}^{(\mathbb{X})}(z) - P_{xy}^{(\mathbb{X})}(\mathcal{B}_{\mathfrak{N}/100}(\{x, y\} \cup \mathbb{X}, \tilde{\mathcal{G}}), z, m_{\text{sc}}(z))| \lesssim N^{-\mathfrak{b}}. \quad (5.7) \quad \{\mathbf{eq:local_la}\}$$

Moreover, the following holds

$$|\tilde{m}_N(z) - m_N(z)| \lesssim \frac{(d-1)^\ell N^\circ \text{Im}[m_N(z)]}{N\eta}, \quad (5.8) \quad \{\mathbf{e:tmmdiff}\}$$

Proof. The first statement (5.7) follows from our assumption that $\tilde{\mathcal{G}} \in \Omega$ and (3.26) (with $\mathbb{X} = \mathbb{T}$). $\{\mathbf{e:tmmdiff}\}$

For the second statement, we recall the notations from (5.5)

$$\begin{aligned} |\tilde{m}_N(z) - m_N(z)| &= \frac{1}{N} \text{Tr}(\tilde{H} - z)^{-1} - \frac{1}{N} \text{Tr}(H - z)^{-1} = \frac{1}{N} \text{Tr}(\tilde{H} - z)^{-1}(H - \tilde{H})(H - z)^{-1} \\ &= \frac{1}{N} \sum_{\alpha \in [\mu]} \text{Tr}(\tilde{H} - z)^{-1} \xi_\alpha (H - z)^{-1} \\ &\leq \frac{1}{\sqrt{d-1}N} \sum_{i=1}^N \sum_{\alpha \in [\mu]} (|\tilde{G}_{il_\alpha}| + |\tilde{G}_{ia_\alpha}| + |\tilde{G}_{ib_\alpha}| + |\tilde{G}_{ic_\alpha}|)(|G_{a_\alpha i}| + |G_{b_\alpha i}| + |G_{c_\alpha i}| + |G_{l_\alpha i}|) \\ &\lesssim \frac{1}{\sqrt{d-1}N} \sum_{i=1}^N \sum_{\alpha \in [\mu]} (|\tilde{G}_{il_\alpha}|^2 + |\tilde{G}_{ia_\alpha}|^2 + |\tilde{G}_{ib_\alpha}|^2 + |\tilde{G}_{ic_\alpha}|^2 + |G_{a_\alpha i}|^2 + |G_{b_\alpha i}|^2 + |G_{c_\alpha i}|^2 + |G_{l_\alpha i}|^2) \\ &\lesssim \frac{(d-1)^\ell N^\circ \text{Im}[\tilde{m}_N(z)]}{N\eta} + \frac{(d-1)^\ell N^\circ \text{Im}[m_N(z)]}{N\eta} \end{aligned} \quad (5.9) \quad \{\mathbf{e:tmmdiff2}\}$$

where the first two statements is from resolvent identity (2.1); the third and fourth statements follows from (5.5); the fifth statement follows from Cauchy-Schwarz inequality; the sixth statement follows from the Ward identity (2.6), (3.22) and $\mu = O((d-1)^\ell)$. Since $N\eta \geq N^\mathfrak{g} \gg (d-1)^\ell N^\circ$, the claim (5.8) follows from rearranging (5.9). $\square$

6 Concentration via exchangeable pair

In this section, we explain the basic ideas of proving concentration using exchangeable pairs, and discuss how to apply this approach in the setting of random d -regular graphs. $\{\mathbf{s:exchangeab}\}$

6.1 Sum of independent Bernoulli random variables

Let $X_1, X_2, \dots, X_n$ be independent Bernoulli(p_i) random variables and define

$$W = \sum_{i=1}^n X_i, \quad \mu = \mathbb{E}W = \sum_{i=1}^n p_i.$$

In this section, we explain how to use exchangeable pair to show that W concentrates.

Choose $I \sim \text{Unif}\{1, \dots, n\}$ independent of everything and let $X'_I \sim \text{Bernoulli}(p_I)$ be an independent copy. Set

$$W' = W - X_I + X'_I.$$

Then (W, W') is an exchangeable pair and satisfies $|W' - W| \leq 1$. Moreover,

$$\mathbb{E}[W' - W \mid X_1, \dots, X_n] = \frac{1}{n} \sum_{i=1}^n (p_i - X_i) = -\frac{1}{n}(W - \mu) =: -f(W).$$

For any smooth g ,

$$\mathbb{E}[f(W)g(W)] = \mathbb{E}[(W - W')g(W)] = \frac{1}{2}\mathbb{E}[(W - W')(g(W) - g(W'))].$$

Take $g(w) = f^{2p-1}(w)$ and set $m(\theta) = \mathbb{E}[e^{\theta f(W)}]$. Then

$$\begin{aligned} \mathbb{E}[f^{2p}(W)] &= \frac{1}{2}\mathbb{E}[(W - W')(f^{2p-1}(W) - f^{2p-1}(W'))] \\ &\leq \frac{2p-1}{4}\mathbb{E}[(W - W')(f(W) - f(W'))(f^{2p-2}(W) + f^{2p-2}(W'))] \\ &= \frac{2p-1}{2n}\mathbb{E}[(W - W')^2 f^{2p-2}(W)] = \frac{2p-1}{2n}\mathbb{E}[\mathbb{E}[(W - W')^2 \mid W] f^{2p-2}(W)]. \end{aligned} \tag{6.1} \quad \{\{\mathbf{e:moment}\}\}$$

We can then bound the inner expectation as

$$\mathbb{E}[(W - W')^2 \mid W] = \frac{1}{n} \sum_{i=1}^n \mathbb{E}[(X_i - X'_i)^2 \mid W] \leq \frac{1}{n} \sum_{i=1}^n \mathbb{E}[(X_i + X'_i) \mid W] = \frac{W + \mu}{n}. \tag{6.2} \quad \{\{\mathbf{e:2moment}\}\}$$

By plugging (6.2) into (6.1), we arrive at the following moment bound

$$\mathbb{E}[f^{2p}(W)] \leq \frac{2p-1}{2n^2}\mathbb{E}[(W + \mu)f^{2p-2}(W)] = \frac{2p-1}{2n}\mathbb{E}[f^{2p-1}(W)] + \frac{(2p-1)\mu}{n^2}\mathbb{E}[f^{2p-2}(W)]. \tag{6.3} \quad \{\{\mathbf{e:moment2}\}\}$$

Then recursively, we can bound the $2p$ -th moment as

$$\mathbb{E}\left[\left(\frac{\sum_i (X_i - p_i)}{n}\right)^{2p}\right] = \mathbb{E}[f^{2p}(W)] \leq \frac{C_p \mu^p}{n^{2p}} \tag{6.4}$$

6.2 Random d -regular graphs

In our setting, we choose a directed edge (i, o) uniformly at random, independently of everything else, and let $\tilde{\mathcal{G}}$ be the graph obtained by locally resampling the neighborhood of the vertex o . Then $(\mathcal{G}, \tilde{\mathcal{G}})$ form an exchangeable pair. In next section we will show that

$$\mathbb{E}[\tilde{G}_{oo}^{(i)} \mid \mathcal{G}] = \mathbb{E}_{\mathbf{S}}[\tilde{G}_{oo}^{(i)}] = Y + \mathcal{E}_{(i,o)}(\mathcal{G}), \tag{6.5}$$

where we recall Y from (4.21), and $\mathcal{E}_{(i,o)}(\mathcal{G})$ is small (here $\mathbb{E}_{\mathbf{S}}$ denotes expectation over the resampling randomness). This implies

$$\mathbb{E}[\tilde{Q} - Q | \mathcal{G}] = \frac{1}{Nd} \sum_{i \sim o} \mathbb{E}_{\mathbf{S}}[\tilde{G}_{oo}^{(i)}] - Q = Y - Q + \mathcal{E}(\mathcal{G}), \quad \mathcal{E}(\mathcal{G}) := \frac{1}{Nd} \sum_{i \sim o} \mathcal{E}_{(i,o)}(\mathcal{G}). \quad (6.6)$$

An argument analogous to the exchangeable-pairs proof of concentration for sums of Bernoulli random variables yields high-moment bounds for $\tilde{Q} - Q$, and hence concentration of $\tilde{Q} - Q$. Although the individual errors $\mathcal{E}_{(i,o)}(\mathcal{G})$ are small—by (3.21) and Lemma 3.9 one obtains bounds of order $O(N^{-b})$ —these are not sufficient to obtain the optimal eigenvalue concentration. For example in Theorem 4.5, near the spectral edge $|z \pm 2| = N^{-2/3+o(1)}$, the optimal error is $N^{-2/3+o(1)}$. To attain this optimal rate, we perform additional local resampling steps to capture and control the fluctuations of the error terms $\mathcal{E}_{(i,o)}(\mathcal{G})$ more precisely. This is outlined in Section 8 and Section 11.

7 Expectation of $Q(z)$.

In this section we take the first step computing the expectation of $Q(z) - Y(z)$.

$$\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)(Q - Y)] = \frac{1}{Nd} \sum_{o,i} \mathbb{E}[A_{oi} \mathbf{1}(\mathcal{G} \in \Omega)(G_{oo}^{(i)} - Y)]. \quad (7.1)$$

For $\mathcal{G} \in \bar{\Omega}$ from Definition 3.1, the number of vertices that do not have a tree neighborhood of radius $\mathfrak{R}$ is at most N^ϵ . We can restrict the righthand side of (7.1) to those $\mathcal{G}$ over vertices o which has radius $\mathfrak{R}$ tree neighborhood:

$$\frac{1}{Nd} \sum_{o,i} \mathbb{E}[A_{oi} \mathbf{1}(\mathcal{G} \in \Omega)(G_{oo}^{(i)} - Y)] = \frac{1}{Nd} \sum_{o,i} \mathbb{E}[I(\{o, i\}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega)(G_{oo}^{(i)} - Y)] + O\left(\frac{1}{N^{1-\epsilon}}\right) \quad (7.2)$$

where

$$I(\{o, i\}, \mathcal{G}) = A_{oi} \mathbf{1}(\mathcal{B}_{\mathfrak{R}}(o, \mathcal{G}) \text{ is a tree}) \quad (7.3)$$

To compute the righthand side of (7.1), conditioned on $I(\{o, i\}, \mathcal{G}) = 1$ we perform a local resampling around $(i, o) \in \mathcal{F}$ using the resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$, where $\mu := d(d-1)^\ell$ is the number of boundary edges of the d -regular tree truncated at depth $\ell+1$. We denote the new graph as $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$, with its corresponding Green's function $\tilde{G}$.

We introduce the following subgraph of $\mathcal{G}$

$$\mathcal{F} = B_\ell(o, \mathcal{G}) \cup \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]} = B_{\ell+1}(o, \mathcal{G}) \cup \{(b_\alpha, c_\alpha)\}_{\alpha \in [\mu]} = (\mathbf{i}, E), \quad (7.4)$$

which contains all the switching edges, see Figure 4. To ensure switching edges are well separated and lie in large tree neighborhoods, we use the following indicator.

$$I(\mathcal{F}, \mathcal{G}) := \prod_{\{x,y\} \in E} A_{xy} \prod_{c \in \{o, c_1, \dots, c_\mu\}} \prod_{x \in \mathcal{B}_\ell(c; \mathcal{G})} \mathbf{1}(\mathcal{B}_{\mathfrak{R}}(x; \mathcal{G}) \text{ is a tree}) \prod_{c \neq c' \in \{o, c_1, \dots, c_\mu\}} \mathbf{1}(\text{dist}_{\mathcal{G}}(c, c') \geq 3\mathfrak{R}). \quad (7.5)$$

Condition on $I(\mathcal{F}, \mathcal{G}) = 1$. Ignoring vertex labels, the graph $\mathcal{F}$ is simply a forest consisting of a d -regular tree truncated at depth $\ell+1$ together with μ disjoint edges. Thus the averaging over the edge (i, o) in (7.1), together with the randomness in the resampling data $\{(b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$, can be viewed as an average over embeddings of $\mathcal{F}$ into $\mathcal{G}$.

To formalize this, let $\mathcal{F} = (V, E)$ denote the *template* whose vertices V are formal symbols, consisting of the truncated d -regular tree of depth $\ell+1$ and the μ disjoint edges. The vertices become concrete vertices of $\mathcal{G}$ only after an embedding is chosen. An embedding is specified by an assignment $\mathbf{i} \in [\mu]^{|\mathcal{F}|}$; different choices of $\mathbf{i}$ yield different embeddings. Once an embedding $\mathbf{i}$ is fixed, we write $\mathcal{F} = (\mathbf{i}, E)$, and the indicator $I(\mathcal{F}, \mathcal{G})$ in (7.5) is well defined.

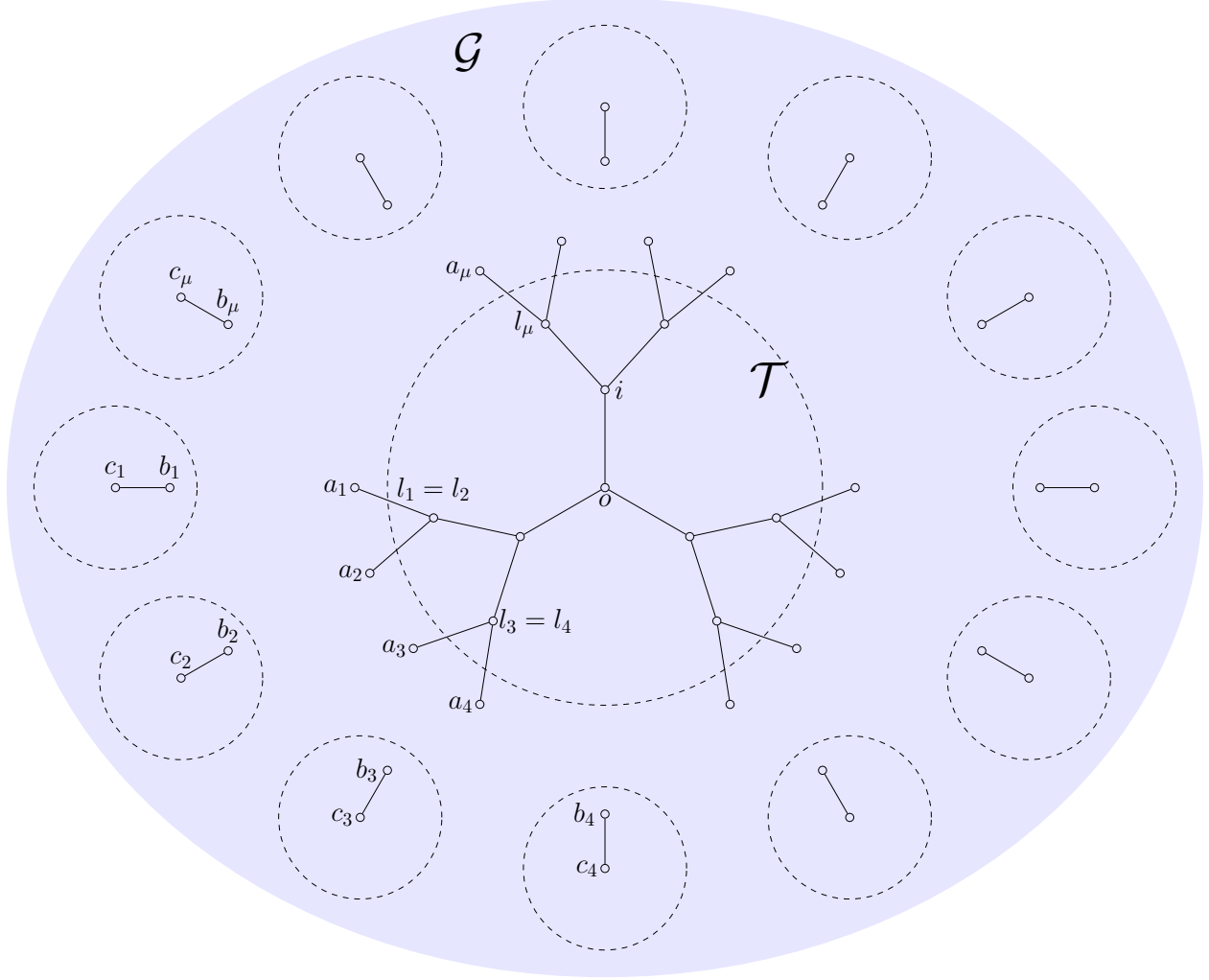


Figure 4: We view $\mathcal{F}$ (corresponding to the solid edges) as an embedded subgraph of $\mathcal{G}$, which contains all the switching edges. The indicator $I(\mathcal{F}, \mathcal{G})$ signifies that $o, c_1, c_2, \dots, c_\mu$ have tree neighborhoods and are well separated.

{fig:FG}

{c:resample}

Claim 7.1. We perform a local resampling around vertex o , and denote the resampled graph by $\tilde{\mathcal{G}}$, then

$$\begin{aligned} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)(Q - Y)] &= \frac{1}{Nd} \sum_{o,i} \mathbb{E} \left[A_{oi} \mathbf{1}(\mathcal{G} \in \Omega) (G_{oo}^{(i)} - Y) \right] \\ &= \frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \right] + O(N^{-b/2} \mathbb{E}[\Psi]), \end{aligned} \quad (7.6) \quad \{\mathbf{e:maint02}\}$$

where the normalization constant $Z_{\mathcal{F}}$ is not random, and for any $\mathcal{G} \in \Omega$, it satisfies

$$\sum_{\mathbf{i}} I(\mathcal{F}, \mathcal{G}) = Z_{\mathcal{F}} \left(1 + O \left(\frac{1}{N^{1-2\epsilon}} \right) \right), \quad (7.7) \quad \{\mathbf{e:ZF}\}$$

The relation (7.7) follows from the fact that most vertices of $\mathcal{G}$ has large tree neighborhood, so the number of “good” embeddings of $\mathcal{F}$ into $\mathcal{G}$ concentrates.

If we temporarily ignore the indicator and the averaging over embeddings, the claim reduces to the simple symmetry

$$\mathbb{E}\left[A_{oi}(G_{oo}^{(i)} - Y)\right] = \mathbb{E}\left[A_{oi}(\tilde{G}_{oo}^{(i)} - Y)\right],$$

which follows from that $(\mathcal{G}, \tilde{\mathcal{G}})$ are identically distributed under the local resampling. The indicator $I(\mathcal{F}, \mathcal{G})$ just restrict attention to “good” placements where we have good estimates for the Green’s function.

In the following proposition, we show that the expectation (7.6) breaks down into an $O(1)$ -weighted sum of terms in the same form, in the following sense.

Definition 7.2. We say $\mathcal{U}$ is an $O(1)$ -weighted sum of (finitely many) terms in the set $\mathcal{R}$, if

$$\mathcal{U} = \sum_{j \geq 1} \mathfrak{c}_j R_j, \quad R_j \in \mathcal{R},$$

and the total weights $\sum_{j \geq 1} |\mathfrak{c}_j| = O(1)$.

Proposition 7.3. We recall $\mathcal{F} = (\mathbf{i}, E)$ from (7.4), and view it as embedded in $\mathcal{G}$ with vertices given by $\mathbf{i}$. Then

$$\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \right] = I_1 + I_2 + \mathcal{E}, \quad (7.8)$$

where $|\mathcal{E}| = O((d-1)^{2\ell} \mathbb{E}[\Psi])$, and

1. I_1 is an $O(1)$ -weighted sum of terms of the following form

$$\frac{1}{(d-1)^{\mathfrak{q}\ell/2} Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} [\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) (G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})} - Y) (G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})} - Q)], \quad (7.9)$$

where $\mathfrak{q} \geq 0$;

2. I_2 is an $O(1)$ -weighted sum of terms of the following form

$$\frac{(d-1)^{3r\ell}}{(d-1)^{\mathfrak{q}\ell/2} Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} [\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) (G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})} - Y) R_{\mathbf{i}}]. \quad (7.10)$$

Here, $\mathfrak{q} \geq 0$ and $r \geq 2$, and the function $R_{\mathbf{i}}$ contains r factors of the form

$$\{(G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})} - Q)\}_{\alpha \in \llbracket \mu \rrbracket}, \quad \{G_{c_{\alpha}c_{\beta}}^{(b_{\alpha}b_{\beta})}, G_{b_{\alpha}b_{\beta}}\}_{\alpha \neq \beta \in \llbracket \mu \rrbracket}, \quad (Q - m_{\text{sc}}(z)). \quad (7.11)$$

7.1 Switching using the Schur complement formula

To prove Proposition 7.3, we need to rewrite $\tilde{G}_{oo}^{(i)} - Y$ which depends on the Green’s functions of the switched graph $\tilde{\mathcal{G}}$, in terms of the Green’s functions of the original graph $\mathcal{G}$, see Figure 3. This is achieved as follows.

Lemma 7.4. Assume that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}, \mathcal{G}) = 1$ from (7.5), and define the index set $\mathbf{A}_i := \{\alpha \in \llbracket \mu \rrbracket : \text{dist}_{\mathcal{T}}(i, l_{\alpha}) = \ell + 1\}$, see Figure 5. The following holds with overwhelmingly high probability over Z : $\tilde{G}_{oo}^{(i)} - Y$ can be rewritten as a weighted sum

$$\tilde{G}_{oo}^{(i)} - Y = \frac{m_{\text{sc}}(z)^{2\ell+2}}{(d-1)^{\ell+1}} \sum_{\alpha \in \mathbf{A}_i} (G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})} - Q) + \frac{m_{\text{sc}}(z)^{2\ell+2}}{(d-1)^{\ell+1}} \sum_{\alpha \neq \beta \in \mathbf{A}_i} G_{c_{\alpha}c_{\beta}}^{(b_{\alpha}b_{\beta})} + \mathcal{U} + \mathcal{E}. \quad (7.12)$$

where $\mathcal{U}$ is an $O(1)$ -weighted sum of terms in the form $(d-1)^{3(r-1)\ell} R_r$, where $r \geq 2$, and R_r contains r factors of the form

$$(G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})}(z) - Q(z)), \quad G_{c_{\alpha}c_{\beta}}^{(b_{\alpha}b_{\beta})}(z), \quad Q(z) - m_{\text{sc}}(z), \quad \alpha \neq \beta \in \llbracket \mu \rrbracket, \quad (7.13)$$

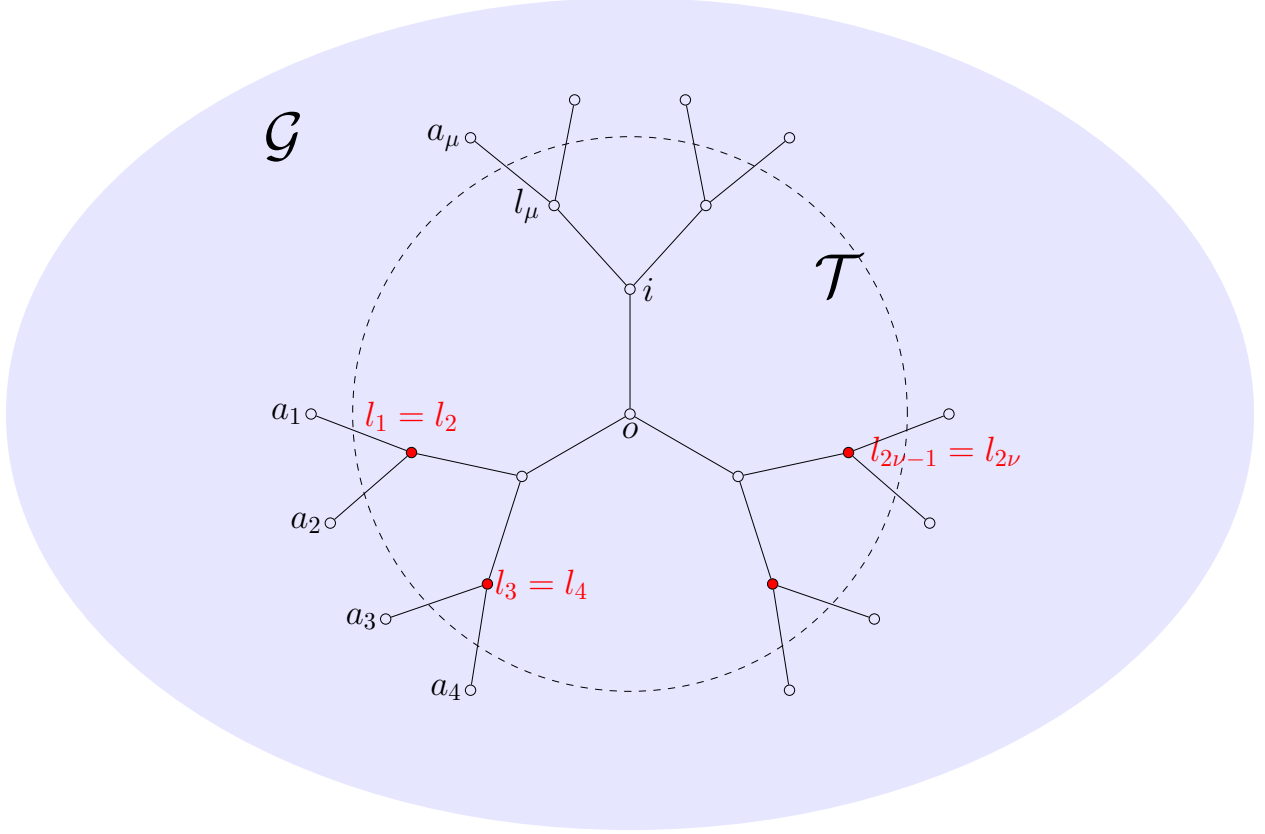


Figure 5: The index set $\mathbf{A}_i := \{\alpha \in \llbracket \mu \rrbracket : \text{dist}_{\mathcal{T}}(i, l_\alpha) = \ell + 1\}$ is given by $\{1, 2, \dots, 2\nu\}$, corresponding to these red nodes.

{fig: Ai}

and at least one of them is $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$ or $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$; and the error $\mathcal{E}$ is given by

$$\begin{aligned} \mathcal{E} &= \frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1}} \sum_{\alpha, \beta \in \mathbf{A}_i} (\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}) + O\left(\frac{1}{N^{\mathfrak{b}/2}} \sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + \frac{1}{N^2}\right) \\ &= O\left(\sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + \frac{1}{N^2}\right). \end{aligned} \quad (7.14) \quad \{\mathbf{e: defCE}\}$$

Remark 7.5. The expansion (7.12) decomposes $\tilde{G}_{oo}^{(i)} - Y$ into two leading contributions and a higher-order remainder $\mathcal{U}$, which together comprise weighted sums of terms built from the factors in (7.13). By Lemma 3.9, each factor in (7.13) is bounded by $N^{-\mathfrak{b}}$ with high probability. Consequently, any product R_r of r such factors satisfies $|R_r| \lesssim N^{-\mathfrak{b}r}$ with high probability. The term $\mathcal{U}$ collects an $O(1)$ -weighted sum of contributions of the form $(d-1)^{3(r-1)\ell} R_r$ with $r \geq 2$, so that

$$(d-1)^{3(r-1)\ell} |R_r| \lesssim (d-1)^{3(r-1)\ell} N^{-\mathfrak{b}r}.$$

Therefore, every contribution to $\mathcal{U}$ is higher-order (by at least one extra factor $N^{-\mathfrak{b}}$) compared to the two leading sums in (7.12).

Remark 7.6. More explicitly, the term $\mathcal{U}$ in (7.12) is given by

$$\begin{aligned}
\mathcal{U} = & \frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+2}} \sum_{\alpha \in \mathbf{A}_i} L_{l_\alpha l_\alpha}^{(i)} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)^2 + \frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+2}} \sum_{\substack{\alpha \in \mathbf{A}_i, \beta \in [\mu] \\ \alpha \neq \beta}} (L_{l_\beta l_\beta}^{(i)} + L_{l_\alpha l_\beta}^{(i)}) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2 \\
& + \sum_{\alpha \neq \beta \in [\mu]} \mathbf{c}_1(\alpha, \beta) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) (G_{c_\beta c_\beta}^{(b_\beta)} - Q) + \sum_{\substack{\alpha \in [\mu] \\ \alpha' \neq \beta' \in [\mu]}} \mathbf{c}_2(\alpha, \alpha', \beta') (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})} \\
& + \sum_{\substack{\alpha \neq \beta \in [\mu], \alpha' \neq \beta' \in [\mu] \\ \{\alpha, \beta\} \neq \{\alpha', \beta'\}}} \mathbf{c}_3(\alpha, \beta, \alpha', \beta') G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})} + \sum_{\alpha \in [\mu]} \mathbf{c}_4(\alpha) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) (Q - m_{\text{sc}}(z)) \\
& + \sum_{\alpha \neq \beta \in [\mu]} \mathbf{c}_5(\alpha, \beta) G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} (Q - m_{\text{sc}}(z)) + \text{terms of the form } \{(d-1)^{3(r-1)\ell} R_r\}_{r \geq 3},
\end{aligned} \tag{7.15} \quad \{\mathbf{e}:\mathbf{Uterm}\}$$

where the total sum of the coefficients $\mathbf{c}_1, \mathbf{c}_2, \dots, \mathbf{c}_5$ is bounded by $O(1)$, and

$$L^{(i)} = P^{(i)}(\mathcal{T}, z, m_{\text{sc}}(z)) = \frac{1}{H_{\mathbb{T}}^{(i)} - z - m_{\text{sc}}(z)\mathbb{I}^\partial}, \quad \mathbb{I}_{xy}^\partial = \mathbf{1}(\text{dist}_{\mathcal{T}}(x, o) = \ell) \delta_{xy}, \tag{7.16} \quad \{\mathbf{e}:\mathbf{2defPi}\}$$

which is the Green's function of the $(d-1)$ -ary tree.

We start with the Schur complement formulas which will be used to prove Lemma 7.4. Let $\tilde{B}$ be the normalized adjacency matrix of the directed edges $\{(c_\alpha, l_\alpha)\}_{\alpha \in [\mu]}$. Then the adjacency matrices $\tilde{H}^{(i)}$ is in the block form

$$\tilde{H}^{(i)} = \begin{bmatrix} H_{\mathbb{T}}^{(i)} & \tilde{B}^\top \\ \tilde{B} & \tilde{H}_{\mathbb{T}^c} \end{bmatrix}.$$

We also denote the Green's function of $\mathcal{G}^{(\mathbb{T})}$ and $\tilde{\mathcal{G}}^{(\mathbb{T})}$ as $G^{(\mathbb{T})}$ and $\tilde{G}^{(\mathbb{T})}$ respectively.

We collect some estimates below, which will be used later. Recall from Proposition 3.3, for $x, y \in \mathbb{T} \setminus \{i\}$, $|L_{xy}^{(i)}| \lesssim (d-1)^{-\text{dist}_{\mathcal{T}}(x,y)/2}$. It follows that

$$\begin{aligned}
\sum_{x \in \mathbb{T} \setminus \{i\}} |L_{ox}^{(i)}| & \lesssim \sum_{r=0}^{\ell} (d-1)^{r/2} \lesssim (d-1)^{\ell/2}, \\
\sum_{x, y \in \mathbb{T} \setminus \{i\}} |L_{xy}^{(i)}| & \lesssim \sum_{r=0}^{\ell} (d-1)^{\ell-r} \left(\sum_{r'=0}^r (d-1)^{r'/2} + \sum_{r'=r+1}^{2\ell-r} (d-1)^{r/2} \right) \lesssim \ell (d-1)^\ell,
\end{aligned} \tag{7.17} \quad \{\mathbf{e}:\mathbf{sum_Pbound}\}$$

where for the first sum, we used that $|\{x \in \mathbb{T} : \text{dist}_{\mathcal{T}}(o, x) = r\}| = O((d-1)^r)$ for $0 \leq r \leq \ell$. For the second sum, we consider $\{x \in \mathbb{T} : \text{dist}_{\mathcal{T}}(o, x) = \ell - r\}$. Note that there are $O((d-1)^{\ell-r})$ such vertices. Given such x , for any $y \in \mathbb{T}$, $0 \leq \text{dist}_{\mathcal{T}}(x, y) \leq 2\ell - r$. There are two cases depending on $\text{dist}_{\mathcal{T}}(x, y) = r'$. If $0 \leq r' \leq r$, we have $|\{y \in \mathbb{T} : \text{dist}_{\mathcal{T}}(x, y) = r'\}| = O((d-1)^{r'})$, and if $r+1 \leq r' \leq 2\ell - r$ we have $|\{y \in \mathbb{T} : \text{dist}_{\mathcal{T}}(x, y) = r'\}| = O((d-1)^{(r+r')/2})$. The second statement in (7.17) follows by summing over r, r' and using $|L_{xy}^{(i)}| \lesssim (d-1)^{-r'/2}$.

Proof of Lemma 7.4. Thanks to the Schur complement formula (2.4), we have

$$\tilde{G}_{oo}^{(i)} = \left(\frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top \tilde{G}^{(\mathbb{T})} \tilde{B}} \right)_{oo}.$$

Since o has a radius $\mathfrak{R}$ tree neighborhood, $H_{\mathbb{T}}^{(i)}$ is the normalized adjacency matrix of a truncated $(d-1)$ -ary tree, and $L^{(i)}$ from (7.16) agrees with the Green's function of $(d-1)$ -ary tree (see (3.3)),

$$m_{\text{sc}}(z) = L_{oo}^{(i)} = P_{oo}^{(i)}(\mathcal{T}, z, m_{\text{sc}}(z)) = \left(\frac{1}{H_{\mathbb{T}}^{(i)} - z - m_{\text{sc}}(z)\mathbb{I}^\partial} \right)_{oo} = \left(\frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top m_{\text{sc}}(z)\tilde{B}} \right)_{oo},$$

By taking the difference of the two above expressions, we have

$$\tilde{G}_{oo}^{(i)} - m_{\text{sc}}(z) = \left(\frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top m_{\text{sc}}(z) \tilde{B} - \mathcal{D}} - \frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top m_{\text{sc}}(z) \tilde{B}} \right)_{oo},$$

where

$$\mathcal{D} = \tilde{B}^\top (Q - m_{\text{sc}}(z)) \tilde{B} + \mathcal{D}_1, \quad \mathcal{D}_1 = \tilde{B}^\top (\tilde{G}^{(\mathbb{T})} - Q) \tilde{B}, \quad (7.18) \quad \{\mathbf{e}:\text{defcE}\}$$

are matrices indexed by $(\mathbb{T} \setminus \{i\}) \times (\mathbb{T} \setminus \{i\})$.

Thanks to (3.21) and (d:lemma3.9), we have

$$|(\mathcal{D}_1)_{xy}|, |\mathcal{D}_{xy}| \leq N^{-\mathbf{b}}, \text{ for } x, y \in \mathbb{T} \setminus \{i\}.$$

Thus, for some sufficiently large constant $\mathbf{p}$, we have

$$\begin{aligned} \tilde{G}_{oo}^{(i)} - m_{\text{sc}}(z) &= \left(\frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top m_{\text{sc}}(z) \tilde{B} - \mathcal{D}} - \frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top m_{\text{sc}}(z) \tilde{B}} \right)_{oo} \\ &= \left(L^{(i)} \sum_{k=1}^{\mathbf{p}} \left(\mathcal{D} L^{(i)} \right)^k \right)_{oo} + O(N^{-2}). \end{aligned} \quad (7.19) \quad \{\mathbf{eq}:\text{resolven}\}$$

Recall $Y = Y_\ell(Q, z)$ from (3.6) and (d:lemma4.23),

$$Y = \left(\frac{1}{H_{\mathbb{T}}^{(i)} - z - \tilde{B}^\top Q \tilde{B}} \right).$$

By the same argument as in (7.19) we also have that

$$Y - m_{\text{sc}}(z) = \left(L^{(i)} \sum_{k=1}^{\mathbf{p}} \left((\tilde{B}^\top (Q - m_{\text{sc}}(z)) \tilde{B} + t(m - m_d(z)) \mathbb{I}) L^{(i)} \right)^k \right)_{oo} + O(N^{-2}). \quad (7.20) \quad \{\mathbf{e}:\mathbf{YQ-m}\}$$

By taking the difference of (7.19) and (7.20), up to error $O(N^{-2})$, we get that the difference $\tilde{G}_{oo}^{(i)} - Y$ is given as

$$\begin{aligned} &(L^{(i)} \mathcal{D}_1 L^{(i)})_{oo} + \left(L^{(i)} \sum_{k=2}^{\mathbf{p}} \left(\mathcal{D} L^{(i)} \right)^k \right)_{oo} \\ &\quad - \left(L^{(i)} \sum_{k=2}^{\mathbf{p}} \left((\tilde{B}^\top (Q - m_{\text{sc}}(z)) \tilde{B} L^{(i)} \right)^k \right)_{oo}. \end{aligned} \quad (7.21) \quad \{\mathbf{e}:\text{GooY}\}$$

If $\alpha \in \mathbf{A}_i$, then $\text{dist}_{\mathcal{T}}(i, l_\alpha) = \ell + 1$, and Proposition 3.3 gives

$$L_{ol_\alpha}^{(i)} = m_{\text{sc}}(z) \left(-\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{T}}(o, l_\alpha)}, \quad |L_{ol_\alpha}^{(i)}| \lesssim (d-1)^{-\text{dist}_{\mathcal{T}}(o, l_\alpha)/2} = (d-1)^{-\ell/2}. \quad (7.22) \quad \{\mathbf{e}:\text{Piol}\}$$

Otherwise if $\alpha \in \llbracket \mu \rrbracket \setminus \mathbf{A}_i$, then o, l_α are in different connected components of $\mathcal{T}^{(i)}$, and $|L_{ol_\alpha}^{(i)}| = 0$. Thus the first term in (7.21) can be computed as,

$$(L^{(i)} \mathcal{D}_1 L^{(i)})_{oo} = \frac{m_{\text{sc}}(z)^{2\ell+2}}{(d-1)^{\ell+1}} \sum_{\alpha \in \mathbf{A}_i} (\tilde{G}_{c_\alpha c_\alpha}^{(\mathbb{T})} - Q) + \frac{m_{\text{sc}}(z)^{2\ell+2}}{(d-1)^{\ell+1}} \sum_{\alpha \neq \beta \in \mathbf{A}_i} \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}, \quad (7.23) \quad \{\mathbf{e}:\text{G-Y2}\}$$

We obtain the first two terms in (7.12), after replacing $\tilde{G}_{c_\alpha c_\alpha}^{(\mathbb{T})} - Q, \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}$ in (7.23) by $G_{c_\alpha c_\alpha}^{(b_\alpha)}, G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$. We collect the difference in the error term $\mathcal{E}$ (as in (7.14)).

For the terms $k \geq 2$, in general for any matrix V defined on $(\mathbb{T} \setminus \{i\}) \times (\mathbb{T} \setminus \{i\})$, $L^{(i)}(VL^{(i)})^k$ is given as a sum of terms in the following form

$$\sum_{x_1, x_2, \dots, x_{2k} \in \mathbb{T} \setminus \{i\}} L_{ox_1}^{(i)} V_{x_1 x_2} L_{x_2 x_3}^{(i)} V_{x_3 x_4} L_{x_4 x_5}^{(i)} \cdots V_{x_{2k-1} x_{2k}} L_{x_{2k} o}^{(i)}. \quad (7.24) \quad \{\mathbf{e:PUP}\}$$

We can reorganize (7.24) in the following way

$$\begin{aligned} & \sum_{x_1, x_2, \dots, x_{2k} \in \mathbb{T} \setminus \{i\}} L_{ox_1}^{(i)} L_{x_2 x_3}^{(i)} \cdots L_{x_{2k} o}^{(i)} V_{x_1 x_2} V_{x_3 x_4} \cdots V_{x_{2k-1} x_{2k}} \\ &= (d-1)^{3(k-1)\ell} \sum_{x_1, x_2, \dots, x_{2k} \in \mathbb{T} \setminus \{i\}} (d-1)^{-3(k-1)\ell} L_{ox_1}^{(i)} L_{x_2 x_3}^{(i)} \cdots L_{x_{2k} o}^{(i)} V_{x_1 x_2} V_{x_3 x_4} \cdots V_{x_{2k-1} x_{2k}} \\ &=: (d-1)^{3(k-1)\ell} \sum_{x_1, x_2, \dots, x_{2k} \in \mathbb{T} \setminus \{i\}} \mathbf{c}_{\mathbf{x}} V_{x_1 x_2} V_{x_3 x_4} \cdots V_{x_{2k-1} x_{2k}}, \end{aligned} \quad (7.25) \quad \{\mathbf{e:totalsum0}\}$$

where the weights $\mathbf{c}_{\mathbf{x}} = (d-1)^{-3(k-1)\ell} L_{ox_1}^{(i)} \cdots L_{x_{2k} o}^{(i)}$, and the total weights are bounded as

$$\begin{aligned} \sum_{x_1, x_2, \dots, x_{2k} \in \mathbb{T} \setminus \{i\}} |\mathbf{c}_{\mathbf{x}}| &= (d-1)^{-3(k-1)\ell} \sum_{x_1 \in \mathbb{T} \setminus \{i\}} |L_{ox_1}^{(i)}| \sum_{x_2, x_3 \in \mathbb{T} \setminus \{i\}} |L_{x_2 x_3}^{(i)}| \cdots \sum_{x_{2k} \in \mathbb{T} \setminus \{i\}} |L_{x_{2k} o}^{(i)}| \\ &\lesssim (d-1)^{-3(k-1)\ell} \ell^{k-1} (d-1)^{k\ell} = (d-1)^{-(2k-3)\ell} \ell^{k-1} \lesssim 1, \end{aligned}$$

where to get the second line we used (7.17); in the total sum inequality, we used that $k \geq 2$.

To compute the difference for $k \geq 2$ in (7.21), we consider two possible forms for V : $V = \tilde{B}^\top (Q - m_{\text{sc}}(z)) \tilde{B}$ or $V = \tilde{B}^\top (Q - m_{\text{sc}}(z)) \tilde{B} + \mathcal{D}_1$. As discussed above (see (7.25)), terms in (7.21) with $k \geq 2$ break down to an $O(1)$ -weighted sum of terms in the form $(d-1)^{3(k-1)\ell} \tilde{R}_k$. Here $\tilde{R}_k$ is a product of k factors, each taking one of the following:

$$(\tilde{G}_{c_\alpha c_\alpha}^{(\mathbb{T})} - Q), \quad \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}, \quad (Q - m_{\text{sc}}(z)), \quad \alpha \neq \beta \in [\mu].$$

Moreover, $\tilde{R}_k$ contains at least one factor of the form $\{\tilde{G}_{c_\alpha c_\alpha}^{(\mathbb{T})} - Q, \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}\}_{\alpha \neq \beta \in [\mu]}$ (arising from $\mathcal{D}_1$). Otherwise, the terms from the difference in (7.21) cancel out.

For each $\tilde{R}_k$ terms, we get R_k by replacing $\tilde{G}_{c_\alpha c_\alpha}^{(\mathbb{T})} - Q, \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}$ with $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q, G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$, respectively. As $k \geq 2$ and each factor of $R_k, \tilde{R}_k$ is bounded by N^{-b} with overwhelmingly high probability over Z , the replacement error is bounded by

$$|\tilde{R}_k - R_k| \lesssim \frac{1}{N^{b/2}} \sum_{\alpha, \beta \in [\mu]} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}|.$$

We collect the above error in $\mathcal{E}$ (as in (7.14)).

We denote the $O(1)$ -weighted sum of terms in the form $(d-1)^{3(k-1)\ell} R_k$ as $\mathcal{U}$. This finishes the proof of Lemma 7.4.

□

7.2 Proof of Proposition 7.3

By plugging (7.12) into (7.8), we get the following four terms

$$\frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1}} \sum_{\alpha \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q) \right], \quad (7.26) \quad \{\mathbf{e}:\mathbf{e1}\}$$

$$\frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1}} \sum_{\alpha \neq \beta \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})} \right], \quad (7.27) \quad \{\mathbf{e}:\mathbf{e2}\}$$

$$\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} [I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \mathcal{U}], \quad (7.28) \quad \{\mathbf{e}:\mathbf{e3}\}$$

$$\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} [I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \mathcal{E}] \quad (7.29) \quad \{\mathbf{e}:\mathbf{e4}\}$$

First term (7.26). The first term (7.26) is negligible

$$\frac{O(1)}{(d-1)^{\ell}} \sum_{\alpha \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q) \right] = O(N^{-b/2} \mathbb{E}[\Psi]). \quad (7.30) \quad \{\mathbf{e}:\mathbf{ftt1}\}$$

If we temporarily ignore the indicator and the averaging over embeddings, the claim (7.30) reflects the definition of Q

$$\frac{1}{Nd} \sum_{b_{\alpha}, c_{\alpha} \in [N]} A_{c_{\alpha} b_{\alpha}} (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q) = 0. \quad (7.31) \quad \{\mathbf{e}:\mathbf{core1}\}$$

which follows from that $(\mathcal{G}, \tilde{\mathcal{G}})$ are identically distributed under the local resampling. In (7.30), for the average over embeddings, we can first sum over the indices b_{α}, c_{α} using (7.31), and then average over other indices $\mathbf{i} \setminus \{b_{\alpha}, c_{\alpha}\}$.

Second term (7.27). We have the following estimate for the second term (7.27)

$$\frac{O(1)}{(d-1)^{\ell}} \sum_{\alpha \neq \beta \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})} \right] \quad (7.32) \quad \{\mathbf{e}:\mathbf{ftt2}\}$$

$$= \frac{O(1)}{(d-1)^{\ell}} \sum_{\alpha \neq \beta \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) G_{b_{\alpha} b_{\beta}} (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q) (G_{c_{\beta} c_{\beta}}^{(b_{\beta})} - Q) \right] + O(N^{-b/4} \mathbb{E}[\Psi]) \quad (7.33) \quad \{\mathbf{e}:\mathbf{ftt2}\}$$

Then we can further change $(G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q)$ in (7.32) to $(G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Y)$ and get

$$\begin{aligned} & \frac{O(1)}{(d-1)^{\ell}} \sum_{\alpha \neq \beta \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Y) \times (G_{b_{\alpha} b_{\beta}} (G_{c_{\beta} c_{\beta}}^{(b_{\beta})} - Q)) \right] \\ & + \frac{O(1)}{(d-1)^{\ell}} \sum_{\alpha \neq \beta \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |Y - Q| G_{b_{\alpha} b_{\beta}} (G_{c_{\beta} c_{\beta}}^{(b_{\beta})} - Q) \right] \end{aligned} \quad (7.34) \quad \{\mathbf{e}:\mathbf{chQY}\}$$

The second term on the righthand side of (7.34) can be further bounded as

$$\frac{O(1)}{(d-1)^{\ell}} \sum_{\alpha \neq \beta \in A_i} \sum_{\mathbf{i}} \frac{1}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |Y - Q| N^{-2b} \right] = O \left(N^{-b/4} \mathbb{E}[\Psi] \right). \quad (7.35)$$

If we temporarily ignore the indicator and the averaging over embeddings, the claim (7.32) boils down to compute the following quantity

$$\frac{1}{(Nd)^2} \sum_{b_{\alpha}, c_{\alpha}, b_{\beta}, c_{\beta} \in [N]} A_{c_{\alpha} b_{\alpha}} A_{b_{\beta} c_{\beta}} G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})}. \quad (7.36) \quad \{\mathbf{e}:\mathbf{core2}\}$$

For random d -regular graphs, the adjacency matrices have a trivial eigenvector $(1, 1, \dots, 1)^\top$. As a consequence the row and column sums of the Green's function is small:

$$\sum_{i \in [N]} G_{ij} = \sum_{j \in [N]} G_{ij} = 1/(d/\sqrt{d-1} - z) = O(1). \quad (7.37) \quad \{\mathbf{e:av1}\}$$

Thus if $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ in (7.36) is replaced by $G^{(b_\alpha b_\beta)}$ (without removing the vertices b_α, b_β), then the average is very small $O(1/N)$. To compute (7.36), we need to express $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ back to $G^{(b_\alpha)}, G^{(b_\beta)}, G$ using Schur complement formula by carefully adding vertices b_α, b_β back. And it turns out we have nice leading term, and all other terms are negligible

$$\begin{aligned} & \frac{1}{(Nd)^2} \sum_{b_\alpha, c_\alpha, b_\beta, c_\beta \in [N]} A_{c_\alpha b_\alpha} A_{b_\beta c_\beta} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} \\ &= \frac{1}{(Nd)^2} \sum_{b_\alpha, c_\alpha, b_\beta, c_\beta \in [N]} A_{c_\alpha b_\alpha} A_{b_\beta c_\beta} \frac{G_{b_\alpha b_\beta}}{d-1} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)(G_{c_\beta c_\beta}^{(b_\beta)} - Q) + \text{"negligible error"} \end{aligned} \quad (7.38) \quad \{\mathbf{e:core21}\}$$

In (7.30), for the average over embeddings, we can first sum over the indices $b_\alpha, c_\alpha, b_\beta, c_\beta$ using (7.38), and then average over other indices $\mathbf{i} \setminus \{b_\alpha, c_\alpha, b_\beta, c_\beta\}$.

Proof of (7.38). For simplicity of notation, we write $b_\alpha, c_\alpha, b_\beta, c_\beta$ as b, c, b', c' . First we notice that using (7.37), the average of Green's functions with only one vertex removed is also small. More precisely, the Schur complement formula (2.5) gives

$$\frac{1}{Nd} \sum_{b \sim c} G_{cc'}^{(b')} = \frac{1}{Nd} \sum_{c \sim b} \left(G_{cc'} - \frac{G_{cb'} G_{b'c'}}{G_{b'b'}} \right) = O\left(\frac{1}{N}\right), \quad \frac{1}{Nd} \sum_{b \sim c} G_{bc'}^{(b')} = O\left(\frac{1}{N}\right). \quad (7.39) \quad \{\mathbf{e:rGbs2}\}$$

To show (7.38), then use the Schur complement formula (2.5), we have

$$\begin{aligned} G_{cc'}^{(bb')} &= G_{cc'}^{(b')} + (G^{(bb')} H)_{cb} G_{bc'}^{(b')} = G_{cc'}^{(b')} + \frac{G_{bc'}^{(b')}}{\sqrt{d-1}} \sum_{x \sim b} G_{cx}^{(bb')} \\ &= G_{cc'}^{(b')} + \frac{G_{bc'}^{(b')}}{\sqrt{d-1}} \sum_{x \sim b} \left(G_{cx}^{(b)} - \frac{G_{cb'}^{(b)} G_{b'x}^{(b)}}{G_{b'b'}} \right) = G_{cc'}^{(b')} + \frac{G_{bc'}^{(b')}}{\sqrt{d-1}} \sum_{x \sim b} G_{cx}^{(b)} - \sum_{x \sim b} \frac{G_{bc'}^{(b')} G_{cb'}^{(b)} G_{b'x}^{(b)}}{\sqrt{d-1} G_{b'b'}}. \end{aligned} \quad (7.40) \quad \{\mathbf{e:rGccbb1}\}$$

If we average over the edges (b, c) , the first term on the righthand side of (7.40) is small by (7.39); for the second term, by the same reasoning, we can replace $\sum_{x \sim b} G_{cx}^{(b)}$ by $\sum_{x \sim b} G_{cx}^{(b)} - Q$; for the last term, we can bound it by the Ward identity. More precisely, (3.26) implies that $|G_{b'x}^{(b)}| \lesssim N^{-b}$ for $b \sim x$. Then (3.27) implies

$$\begin{aligned} \frac{A_{b'c'}}{Nd} \sum_{b \sim c} \sum_{x \sim b} \left| \frac{G_{bc'}^{(b')} G_{cb'}^{(b)} G_{b'x}^{(b)}}{G_{b'b'}} \right| &\lesssim \frac{A_{b'c'} N^{-b}}{N} \sum_{b \sim c} |G_{bc'}^{(b')} G_{cb'}^{(b)}| \\ &\lesssim \frac{A_{b'c'} N^{-b}}{N} \sum_{b \sim c} (|G_{bc'}^{(b')}|^2 + |G_{cb'}^{(b)}|^2) \lesssim N^{-b/2} A_{b'c'} \Phi. \end{aligned} \quad (7.41) \quad \{\mathbf{e:rwarduse1}\}$$

Thus averaging over the edges (b, c) for (7.40) we conclude that

$$\frac{A_{b'c'}}{Nd} \sum_{b \sim c} G_{cc'}^{(bb')} = \frac{A_{b'c'}}{Nd} \sum_{b \sim c} \left(\frac{G_{bc'}^{(b')}}{\sqrt{d-1}} \left(\sum_{x \sim b} G_{cx}^{(b)} - Q \right) + O(N^{-b/2} \Phi) \right). \quad (7.42) \quad \{\mathbf{e:rGbbcc0}\}$$

We can then average over edges (b', c') in (7.42). Similarly to (7.40), we can first replace $G_{bc'}^{(b')}$ as

$$G_{bc'}^{(b')} = G_{bc'} + G_{bb'}(HG^{(b')})_{b'c'} = G_{bc'} + \frac{G_{bb'}}{\sqrt{d-1}} \sum_{x \sim b'} G_{c'x}^{(b')}, \quad (7.43) \quad \{\mathbf{e}: \text{Gccbb2}\}$$

and conclude

$$\begin{aligned} \frac{1}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} G_{cc'}^{(bb')} &= \frac{1}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} \frac{G_{bb'}}{d-1} \left(\sum_{x \sim b} G_{cx}^{(b)} - Q \right) \left(\sum_{x \sim b'} G_{c'x}^{(b')} - Q \right) + O(N^{-b/2} \Phi) \\ &= \frac{1}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} \frac{G_{bb'}}{d-1} (G_{cc}^{(b)} - Q)(G_{c'c'}^{(b')} - Q) \\ &\quad + \frac{N^{-b/2}}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} O \left(|G_{bb'}| \left(\sum_{x \sim b, x \neq c} |G_{cx}^{(b)}| + \sum_{x \sim b', x \neq c'} |G_{c'x}^{(b')}| \right) + \Phi \right), \end{aligned}$$

where for the last equality we bound $|G_{cx}^{(b)}| \lesssim N^{-b}$ for $x \sim b, x \neq c$, and $|G_{c'x}^{(b')}| \lesssim N^{-b}$ for $x \sim b', x \neq c'$ using (3.26). This gives (7.38). $\square$

Third term (7.28). We recall the first few terms of $\mathcal{U}$ from (7.15). We have the following decomposition for the third term in (7.28)

$$\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} [I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \mathcal{U}] = J_1 + J_2 + J_3, \quad (7.44) \quad \{\mathbf{e}: \text{decompose}\}$$

where

$$J_1 = \sum_{\alpha \in A_i} \sum_{\mathbf{i}^+} \frac{m_{\text{sc}}^{2(\ell+1)}(z) L_{l_{\alpha} l_{\alpha}}^{(i)}}{(d-1)^{\ell+2} Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q)^2 \right], \quad (7.45) \quad \{\mathbf{e}: \text{defI1}\}$$

$$J_2 = \sum_{\substack{\alpha \in A_i, \beta \in [\mu] \\ \alpha \neq \beta}} \sum_{\mathbf{i}^+} \frac{m_{\text{sc}}^{2(\ell+1)}(z) (L_{l_{\beta} l_{\beta}}^{(i)} + L_{l_{\alpha} l_{\beta}}^{(i)})}{(d-1)^{\ell+2} Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})})^2 \right]. \quad (7.46) \quad \{\mathbf{e}: \text{defI2}\}$$

And J_3 is an $O(1)$ -weighted sum of terms in the form

$$\sum_{\mathbf{i}} \frac{(d-1)^{3(h-1)}}{Z_{\mathcal{F}}} \mathbb{E} [I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) R_h], \quad (7.47) \quad \{\mathbf{e}: \text{fhatRi03}\}$$

where R_h is a product of h terms in the form (7.11), which are either $G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q$ or $G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})}$. Moreover, either $h \geq 3$, or $h = 2$ and R_h (recall from (7.15)) is one of the following terms

$$\begin{aligned} &(G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q)(G_{c_{\beta} c_{\beta}}^{(b_{\beta})} - Q), \quad (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q)G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})}, \quad G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})} G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})}, \\ &\{G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q, G_{c_{\alpha} c_{\beta}}^{(b_{\alpha} b_{\beta})}\} \times (Q - m_{\text{sc}}(z)), \end{aligned} \quad (7.48) \quad \{\mathbf{e}: \text{Rhform}\}$$

where $\alpha \neq \beta \in [\mu]$, $\alpha' \neq \beta' \in [\mu]$ and $\{\alpha, \beta\} \neq \{\alpha', \beta'\}$.

In the following we discuss the three terms J_1, J_2, J_3 one by one. For J_1 , by the same argument as in (7.34), we can change one copy of $(G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q)$ in (7.45) to $(G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Y)$, and the error is bounded by $O(N^{-b/2} \mathbb{E}[\Psi])$. After such replacement, we get

$$\sum_{\alpha \in A_i} \sum_{\mathbf{i}} \frac{m_{\text{sc}}^{2(\ell+1)}(z) L_{l_{\alpha} l_{\alpha}}^{(i)}}{(d-1)^{\ell+2} Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Y)(G_{c_{\alpha} c_{\alpha}}^{(b_{\alpha})} - Q) \right], \quad (7.49) \quad \{\mathbf{e}: \text{newterm3}\}$$

which is an $O(1)$ -weighted sum of terms in the form (7.9) with $\{e:G_{cc'}\}$.

For J_2 as in (7.46), we have the following estimate

$$|J_2| \lesssim \sum_{\alpha \neq \beta} \frac{1}{(d-1)^\ell Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}|^2 \right] \lesssim (d-1)^{2\ell} \mathbb{E}[\Psi], \quad (7.50) \quad \{\{e:G_{cc'}\}\}$$

so this contribution can be absorbed into the error term $\mathcal{E}$ in (7.8). $\{\{e:G_{cc'}\}\}$

If we temporarily ignore the indicator and the averaging over embeddings, the claim (7.50) boils down to compute the following quantity

$$\frac{1}{(Nd)^2} \sum_{b_\alpha, c_\alpha, b_\beta, c_\beta \in [N]} A_{c_\alpha b_\alpha} A_{b_\beta c_\beta} |G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}|^2 \lesssim \Phi. \quad (7.51) \quad \{\{e:G_{cc'}\}\}$$

Proof of (7.51). Averaging over (7.51) can be bounded by the classical Ward identity (2.6), which gives

$$\frac{1}{N^2} \sum_{i,j} |G_{ij}(z)|^2 = \frac{\text{Im } m_N(z)}{N \text{Im}[z]} \lesssim \Phi(z), \quad (7.52) \quad \{\{eq:rwardex\}\}$$

However, the entry $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ (the Green's function with rows/columns b_α, b_β removed) depends on the choice of these edges, but we can express it in terms of the full Green's function G by the Schur complement formula and bound the average using the Ward identity. For simplicity of notation, we write $b_\alpha, c_\alpha, b_\beta, c_\beta$ as b, c, b', c' .

We start with the Schur complement formula (2.4) $\{\{eq:rwardex\}\}$

$$G_{cc'}^{(bb')} = G_{cc'} - (G(G|_{\{bb'\}})^{-1}G)_{cc'}, \quad (G|_{\{bb'\}})^{-1} = \frac{1}{G_{bb}G_{b'b'} - G_{bb'}^2} \begin{bmatrix} G_{b'b'} & -G_{bb'} \\ -G_{bb'} & G_{bb} \end{bmatrix} \quad (7.53) \quad \{\{e:scf\}\}$$

Thanks to (3.21), the Green's function terms are bounded, and the denominators in (7.53) are bounded away from 0, we can get

$$|G_{cc'}^{(bb')}| \lesssim |G_{bb'}| + |G_{bc'}| + |G_{b'c}| + |G_{cc'}|. \quad (7.54) \quad \{\{e:G_{cc'}^{bb'}\}\}$$

Then the Ward identity bound (7.52) leads to $\{\{e:G_{cc'}^{bb'}\}\}$

$$\frac{1}{(Nd)^2} \sum_{\substack{c \sim b \\ c' \sim b'}} |G_{cc'}^{(bb')}|^2 \lesssim \frac{1}{(Nd)^2} \sum_{\substack{c \sim b \\ c' \sim b'}} I_{cc'} (|G_{bb'}|^2 + |G_{bc'}|^2 + |G_{b'c}|^2 + |G_{cc'}|^2) \lesssim \Phi. \quad (7.55) \quad \{\{e:ccbb\}\}$$

□

Finally, for J_3 , if $h = 2$, R_h is given in (7.48). There are two cases

1. R_h contains a factor $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$, then we can first average over edges (b_α, c_α) as in (7.31). In this case we can bound (7.47) by the same way as for (7.30), and it is bounded by $O(N^{-b/2} \mathbb{E}[\Psi])$
2. R_h contains a factor $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$, then we can first average over edges $(b_\alpha, c_\alpha), (b_\beta, c_\beta)$ as in (7.36). In this case we can estimate (7.47) by the same way as for (7.32), and get

$$\frac{1}{(d-1)} \frac{(d-1)^{3(h+1)\ell}}{(d-1)^{q\ell/2} Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) R'_{h+1} \right] + O(N^{-b/4} \mathbb{E}[\Psi]), \quad (7.56) \quad \{\{e:rhuang\}\}$$

where R'_{h+1} is obtained from R_h by replacing $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ by $G_{b_\alpha b_\beta} (G_{c_\beta c_\beta}^{(b_\beta)} - Q)$, and it contains $h+1$ factors in the form (7.11); see (7.11). This leads to (7.10).

If $h \geq 3$, we recall that R_h contains either $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$ or $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$. There are three cases

1. If R_h contains a factor $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$, we can change it to $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y)$, and the error is bounded by $O(N^{-b/2} \mathbb{E}[\Psi])$.
2. If R_h contains two factors in the form $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$, by Cauchy-Schwarz inequality and similarly to (7.51), we can bound it as

$$\sum_{\mathbf{i}} \frac{(d-1)^{3(h-1)}}{Z_{\mathcal{F}}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) N^{-(h-2)b} |G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}|^2 \right] = O(N^{-b/2} \mathbb{E}[\Psi]), \quad (7.57)$$

3. If R_h does not contain any factor in the form $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$, and contains exactly one factor in the form $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$, then by the same reasoning as in (7.56), up to negligible error we can replace $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ by $(G_{c_\alpha c_\alpha}^{(b_\beta)} - Y) \times (G_{b_\alpha b_\beta} (G_{c_\beta c_\beta}^{(b_\beta)} - Q))$.

All three cases above lead to (7.10).

Fourth term (7.29). We recall $\mathcal{E}$ from (7.14), the last term (7.29) is given by

$$\begin{aligned} & \frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) \mathcal{E} \right] \\ &= \frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) \left(\frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1}} \sum_{\alpha, \beta \in \mathbf{A}_i} (\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}) \right) \right] \\ &+ O \left(\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) \left(\frac{1}{N^{b/2}} \sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + \frac{1}{N^2} \right) \right] \right). \end{aligned} \quad (7.58) \quad \{\mathbf{e}:\mathbf{Eterm1}\}$$

We can express $\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}$ in terms of the Green's function G of the original graph using Schur complement formula. In Lemma 13.3 we show

$$\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} = G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{E}_{\alpha\beta}, \quad (7.59) \quad \{\mathbf{e}:\text{replaceex}\}$$

where

$$\begin{aligned} |\mathcal{E}_{\alpha\beta}| &\lesssim \sum_{\gamma \in \llbracket \mu \rrbracket, x \in \mathcal{N}_\gamma} (|G_{c_\alpha x}^{(\text{TW})}|^2 + |G_{c_\beta x}^{(\text{TW})}|^2) + \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} (|G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2 + |G_{c_\beta b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2) \\ &+ \sum_{x \in \mathbb{T}} (|G_{c_\alpha x}^{(b_\alpha b_\beta)}|^2 + |G_{c_\beta x}^{(b_\alpha b_\beta)}|^2) + N^{-b} \Phi, \end{aligned} \quad (7.60) \quad \{\mathbf{e}:\text{rcEbound}\}$$

and $N_\gamma = \{x \neq c_\gamma : x \sim b_\gamma\} \cup \{a_\gamma\}$.

After averaging over embeddings of $\mathcal{F}$ (i.e., over $\mathbf{i}$) most sums in (7.60) are controlled by (7.52). For instance, averaging over (b_α, c_α) and (b_γ, c_γ) bounds $\mathbb{E}[|G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2]$ via Ward identity.

The delicate case is terms such as $|G_{c_\alpha x}^{(\text{TW})}|^2$ with $x \in \mathcal{N}_\alpha$, where x and c_α are graph-neighbors of a common vertex and the embedding average does not decouple their indices. To handle these, we use a new Ward-type bound **how about call it punctured-vertex Ward bound?**: if i, j are two neighbors of a vertex o (so $o \sim i$ and $o \sim j$), then

$$\mathbb{E} \left[|G_{ij}^{(o)}(z)|^2 \right] \leq N^o \frac{\mathbb{E}[\text{Im } m_N(z)]}{N\eta}, \quad (7.61) \quad \{\mathbf{e}:\mathbf{Gijo-bound}\}$$

proved in Lemma 13.1. The idea is to perform local resampling at o , use the invariance $\mathbb{E}[|G_{ij}^{(o)}|^2] = \mathbb{E}[|\tilde{G}_{ij}^{(o)}|^2]$. We then expand $\tilde{G}_{ij}^{(o)}(z)$ using the Schur complement formula. Crucially, we can bound $\mathbb{E}[|\tilde{G}_{ij}^{(o)}(z)|^2]$, by $\mathbb{E}[|G_{ij}^{(o)}(z)|^2]$ times a small factor, and errors as in (7.52), leading to the desired bound given by the right-hand side of (7.61).

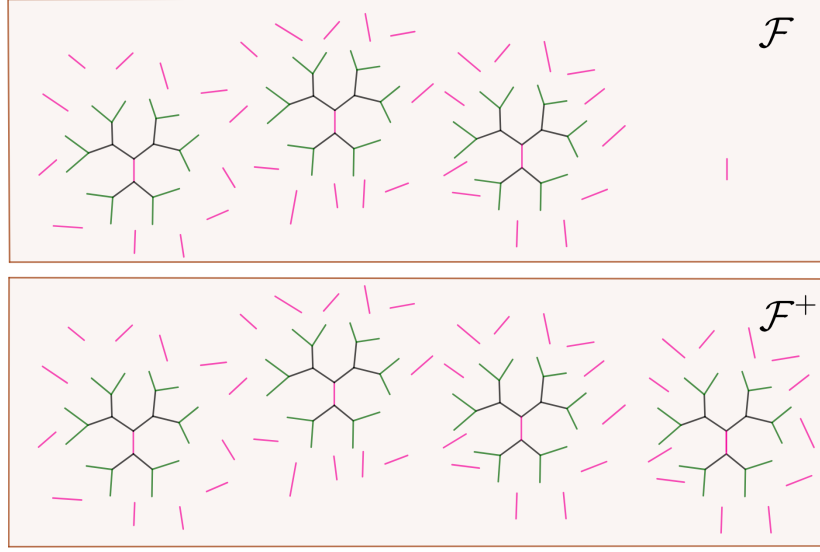


Figure 6: Top Panel: In the forest $\mathcal{F}$, red edges represent core edges $\mathcal{C}$. The used core edges belong to radius- $(\ell + 1)$ balls, while each unused core edge $\mathcal{C}^\circ$ forms its own connected component. Together, the red and green edges constitute the switching edges $\mathcal{K}$. Bottom Panel: We construct $\mathcal{F}^+$ from $\mathcal{F}$ by selecting an unused core edge (the rightmost red edge), expanding it into a radius- $(\ell + 1)$ ball, and adding μ new switching edges. fig:forest

Combining (7.59), (7.60) with Ward's identity (7.52) and the bound (7.61), we obtain

$$\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) \sum_{\alpha, \beta \in \mathbf{A}_i} (\tilde{G}_{c_{\alpha}c_{\beta}}^{(\mathbb{T})} - G_{c_{\alpha}c_{\beta}}^{(b_{\alpha}b_{\beta})}) \right] = O((d-1)^{2\ell} \mathbb{E}[\Psi]),$$

and the second line in (7.58) is $O(N^{-b/4} \mathbb{E}[\Psi])$. See Section 13 for the full statement and proof.

8 Switching edges, the forest and admissible functions

{s:forest}

We recall from Proposition 7.3 that, up to negligible errors,

$$\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)(Q - Y)]$$

decomposes into an $O(1)$ -weighted sum of terms of the form

$$\mathbb{E}[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_{\alpha}c_{\alpha}}^{(b_{\alpha})} - Y) R_{\mathbf{i}}], \quad (8.1) \quad \{\mathbf{e}:R_{\mathbf{i}}\text{first}\}$$

where $\mathcal{F}$ from (7.4) contains all switching edges, and $R_{\mathbf{i}}$ is a product of factors of the form (7.11). To estimate (8.1), we perform a local resampling around the edge (b_{α}, c_{α}) and repeat the procedure from Section 7, which naturally leads to an iterative scheme.

At each iteration, we locally resample the graph and express the Green's function of the switched graph in terms of the original one as in Section 7. Because almost all neighborhoods in $\mathcal{G}$ are acyclic, with high probability the edges involved in these resamplings have large tree neighborhoods and are typically far apart. Hence, we may regard them collectively as forming a forest. In this section, we formalize this iterative procedure using a sequence of forests (see Figure 6), which encode all edges involved in local resamplings.

Fix $\mu := d(d-1)^\ell$, the number of boundary edges of the d -regular tree truncated at depth $\ell+1$. We begin with a two-vertex *template* (an unlabeled graph):

$$\mathcal{F}_0 := (V_0, E_0), \quad V_0 = \{\mathbf{i}_0, \mathbf{o}_0\}, \quad E_0 = \{(\mathbf{i}_0, \mathbf{o}_0)\}.$$

Here $\mathbf{i}_0$ and $\mathbf{o}_0$ are formal symbols (distinguished template vertices) that become concrete vertices of $\mathcal{G}$ only once an embedding of $\mathcal{F}$ into $\mathcal{G}$ is specified.

Initialization. Construct $\mathcal{F}_1$ from $\mathcal{F}_0$ as follows:

- extend the directed edge $(\mathbf{i}_0, \mathbf{o}_0)$ to a truncated d -regular tree $\mathcal{T}_\ell(\mathbf{o}_0)$ rooted at $\mathbf{o}_0$;
- add μ boundary edges $\mathcal{M}'_1 = \{e'_\alpha\}_{\alpha \in [\mu]}$ to $\mathcal{T}_\ell(\mathbf{o}_0)$ to obtain $\mathcal{T}_{\ell+1}(\mathbf{o}_0)$;
- introduce μ new directed edges $\mathcal{M}_1 = \{e_\alpha\}_{\alpha \in [\mu]}$.

Thus,

$$\mathcal{F}_1 = \mathcal{T}_{\ell+1}(\mathbf{o}_0) \cup \mathcal{M}_1, \quad \mathcal{T}_\ell(\mathbf{o}_0) \cup \mathcal{M}'_1 = \mathcal{T}_{\ell+1}(\mathbf{o}_0). \quad (8.2) \quad \{\text{eq:forestde}\}$$

The subgraph $\mathcal{F}$ in (7.4) can be oriented as an embedding of $\mathcal{F}_1$ into $\mathcal{G}$.

General step. Given $\mathcal{F}_s = (V_s, E_s)$ and the sets $\mathcal{M}'_s, \mathcal{M}_s$ created at the previous steps, construct $\mathcal{F}_{s+1}$ by

- choosing a (previously created) core edge $e = (\mathbf{i}_s, \mathbf{o}_s) \in \mathcal{M}_s$ and extending it to $\mathcal{T}_\ell(\mathbf{o}_s)$;
- adding μ boundary edges $\mathcal{M}'_{s+1} = \{e'_\alpha\}_{\alpha \in [\mu]}$ to get $\mathcal{T}_{\ell+1}(\mathbf{o}_s)$;
- adding μ new directed edges $\mathcal{M}_{s+1} = \{e_\alpha\}_{\alpha \in [\mu]}$.

Explicitly,

$$\mathcal{F}_{s+1} = \mathcal{F}_s \cup \mathcal{T}_{\ell+1}(\mathbf{o}_s) \cup \mathcal{M}_{s+1}, \quad \mathcal{T}_\ell(\mathbf{o}_s) \cup \mathcal{M}'_{s+1} = \mathcal{T}_{\ell+1}(\mathbf{o}_s). \quad (8.3) \quad \{\text{eq:forestde}\}$$

We refer to Figure 6 for the construction of $\mathcal{F}_{s+1} = \mathcal{F}^+$ from $\mathcal{F}_s = \mathcal{F}$.

Embedding into $\mathcal{G}$. Local resampling (Section 5) yields an embedding of $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_{s+1}$ into $\mathcal{G}$. Under an embedding we write $\mathcal{F}_s = \mathcal{F}_{\mathbf{i}_s}$ with vertex tuple $\mathbf{i}_s \in [N]^{|V_s|}$; $\mathbf{i}_s \mapsto i_s$ and $\mathbf{o}_s \mapsto o_s$. The neighborhood $\mathcal{T}_\ell(\mathbf{o}_s)$ maps to the ball $B_\ell(o_s; \mathcal{G})$; boundary edges e'_α, e_α map to (ℓ_α, a_α) and (b_α, c_α) in $\mathcal{G}$.

Core and switching edges. Let

$$\mathcal{K}_s := \{(\mathbf{i}_0, \mathbf{o}_0)\} \cup \mathcal{M}_1 \cup \mathcal{M}'_1 \cup \dots \cup \mathcal{M}_s \cup \mathcal{M}'_s$$

be the set of switching edges, and

$$\mathcal{C}_s := \{(\mathbf{i}_0, \mathbf{o}_0)\} \cup \mathcal{M}_1 \cup \dots \cup \mathcal{M}_s$$

the core edges (each component of $\mathcal{F}_s$ contains exactly one core edge). Write the used core edges as $\{(\mathbf{i}_0, \mathbf{o}_0), \dots, (\mathbf{i}_{s-1}, \mathbf{o}_{s-1})\}$ and $\mathcal{C}_s^\circ := \mathcal{C}_s \setminus \{(\mathbf{i}_0, \mathbf{o}_0), \dots, (\mathbf{i}_{s-1}, \mathbf{o}_{s-1})\}$ for the unused ones.

For most arguments we freeze a step and abbreviate

$$\mathcal{F} = \mathcal{F}_s, \quad \mathcal{K} = \mathcal{K}_s, \quad \mathcal{C} = \mathcal{C}_s, \quad \mathcal{C}^\circ = \mathcal{C}_s^\circ, \quad \mathcal{F}^+ = \mathcal{F}_{s+1}, \quad \mathcal{K}^+ = \mathcal{K}_{s+1}, \quad \mathcal{C}^+ = \mathcal{C}_{s+1}, \quad (\mathcal{C}^\circ)^+ = \mathcal{C}_{s+1}^\circ.$$

In general,

$$\mathcal{F} = \bigcup_{e \in \mathcal{C}^\circ} \{e\} \cup \bigcup_{(\mathbf{i}', \mathbf{o}') \in \mathcal{C} \setminus \mathcal{C}^\circ} \mathcal{T}_{\ell+1}(\mathbf{o}'), \quad \mathcal{K} = \mathcal{C} \cup \bigcup_{(\mathbf{i}', \mathbf{o}') \in \mathcal{C} \setminus \mathcal{C}^\circ} (\mathcal{T}_{\ell+1}(\mathbf{o}') \setminus \mathcal{T}_\ell(\mathbf{o}')). \quad (8.4) \quad \{\text{eq:cFtoCF+}\}$$

If we expand an unused core edge $(\mathbf{i}, \mathbf{o}) \in \mathcal{C}^\circ$ by $\mathcal{T}_{\ell+1}(\mathbf{o}) = \mathcal{T}_\ell(\mathbf{o}) \cup \mathcal{M}'$, where $\mathcal{M}' = \{e'_\alpha\}_{\alpha \in [\mu]}$, and add $\mathcal{M} = \{e_\alpha\}_{\alpha \in [\mu]}$, then

$$\mathcal{C}^+ = \mathcal{C} \cup \mathcal{M}, \quad (\mathcal{C}^\circ)^+ = \mathcal{C}^\circ \cup \mathcal{M} \setminus \{(\mathbf{i}, \mathbf{o})\}, \quad \mathcal{F}^+ = \mathcal{F} \cup \mathcal{T}_{\ell+1}(\mathbf{o}) \cup \mathcal{M}, \quad \mathcal{K}^+ = \mathcal{K} \cup \mathcal{M} \cup \mathcal{M}'. \quad (8.5) \quad \{\text{eq:cF++}\}$$

We refer to Figure 6 for $\mathcal{F}$ and $\mathcal{F}^+$.

Indicator for “good” embeddings. For most of this paper we view $\mathcal{F} = (V, E)$ as an embedded (labeled) copy of the template in $\mathcal{G}$; this amounts to choosing a vertex assignment $\mathbf{i} \in \llbracket N \rrbracket^{|V|}$. Different choices of $\mathbf{i}$ yield different embeddings. Under this convention, $\mathcal{K}$, $\mathcal{C}$, and $\mathcal{C}^\circ$ are also regarded as embedded subgraphs of $\mathcal{G}$. With a slight abuse of notation, we write $\mathcal{F} = (\mathbf{i}, E)$ to emphasize that we have fixed an embedding of $\mathcal{F}$ into $\mathcal{G}$. As discussed above, one local resampling step produces a new embedded forest $\mathcal{F}^+ = \mathcal{F}^+(\mathbf{i}^+, E^+) \subset \mathcal{G}$ with vertex tuple $\mathbf{i}^+$.

The forest $\mathcal{F}$ records the edges involved in all previous local resamplings. To ensure these edges are well separated and lie in large tree neighborhoods, we use the following indicator. {def:indicator}

Definition 8.1. Let $\mathcal{F} = (\mathbf{i}, E)$ (as in (8.4)) with core edges $\mathcal{C}$, viewed as a subgraph of a d -regular graph $\mathcal{G}$ with adjacency matrix A . Define $I(\mathcal{F}, \mathcal{G}) = 1$ if and only if: (i) $\mathcal{F}$ is embedded in $\mathcal{G}$; (ii) for every $(b, c) \in \mathcal{C}$ and every $x \in \mathcal{B}_\ell(c; \mathcal{G})$, the ball $\mathcal{B}_\mathfrak{R}(x; \mathcal{G})$ is a tree; and (iii) distinct core edges are at pairwise distance at least $3\mathfrak{R}$. Equivalently,

$$I(\mathcal{F}, \mathcal{G}) := \prod_{\{x, y\} \in E} A_{xy} \prod_{(b, c) \in \mathcal{C}} \prod_{x \in \mathcal{B}_\ell(c; \mathcal{G})} \mathbf{1}(\mathcal{B}_\mathfrak{R}(x; \mathcal{G}) \text{ is a tree}) \prod_{\substack{(b, c), (b', c') \in \mathcal{C} \\ (b, c) \neq (b', c')}} \mathbf{1}(\text{dist}_{\mathcal{G}}(c, c') \geq 3\mathfrak{R}). \quad (8.6) \quad \{\mathbf{f}:\mathbf{def1}\}$$

Fix a good embedding of $\mathcal{F} = (\mathbf{i}, E)$ into $\mathcal{G} \in \Omega$ with $I(\mathcal{F}, \mathcal{G}) = 1$. Then we perform a local resampling around an unused core edge (i, o) . In the following lemma, we show that with high probability with respect to the randomness of $\mathbf{S}$, the randomly selected edges (b_α, c_α) are far away from each other, and have large tree neighborhood. In particular $\tilde{\mathcal{G}} = T_{\mathbf{S}}\mathcal{G} \in \bar{\Omega}$. {lem:configur}

Lemma 8.2. Fix a d -regular graph $\mathcal{G} \in \bar{\Omega}$, and a forest $\mathcal{F} = (\mathbf{i}, E)$ (as in (8.4)) which is a subgraph of $\mathcal{G}$. Assume that $I(\mathcal{F}, \mathcal{G}) = 1$ and $|\mathbf{i}| \leq N^{\epsilon/2}$. We consider the local resampling around an unused core edge $(i, o) \in \mathcal{C}^\circ$, with resampling data $\{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in \llbracket \mu \rrbracket}$. We denote the set of resampling data $\mathbf{F}(\mathcal{G}) \subset \mathbf{S}(\mathcal{G})$ such that the following holds

1. for any $\alpha \neq \beta \in \llbracket \mu \rrbracket$, $\text{dist}_{\mathcal{G}}(\{b_\alpha, c_\alpha\} \cup \mathbf{i}, \{b_\beta, c_\beta\}) \geq 3\mathfrak{R}$;
2. for any $v \in \mathcal{B}_\ell(\{b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}, \mathcal{G})$, the radius $\mathfrak{R}$ neighborhood of v is a tree.

Then $\mathbb{P}_{\mathbf{S}}(\mathbf{F}(\mathcal{G})) \geq 1 - N^{-1+2\epsilon}$ (where $\mathbb{P}_{\mathbf{S}}(\cdot)$ is the probability with respect to the randomness of $\mathbf{S}$ as in Definition 5.3). Also, for $\mathbf{S} \in \mathbf{F}(\mathcal{G})$ the following holds

1. $\mu = d(d-1)^\ell$, $\mathbf{W}_{\mathbf{S}} = \llbracket \mu \rrbracket$ (recall from (5.2)), and $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G}) \in \bar{\Omega}$;
2. $I(\mathcal{F}^+, \mathcal{G}) = 1$ and $I(\mathcal{F}, \tilde{\mathcal{G}}) = 1$.

Proof of Lemma 8.2. We sequentially select (b_α, c_α) uniformly random from $\mathcal{G}^{(\mathbb{T})}$. For any fixed α , we consider all edges that would break the requirements of the lemma. For the first requirement, we have

$$\mathbb{P}_{\mathbf{S}}(\text{dist}_{\mathcal{G}}(\mathbf{i} \cup \bigcup_{1 \leq \beta \leq \alpha-1} \{b_\beta, c_\beta\}, \{b_\alpha, c_\alpha\}) \leq 3\mathfrak{R}) \lesssim N^{-1}(|\mathbf{i}| + 2\mu)d(d-1)^{3\mathfrak{R}} \leq N^{-1+3\epsilon/2}. \quad (8.7) \quad \{\mathbf{f}:\mathbf{small11}\}$$

For the second requirement, we recall that $\mathcal{G} \in \bar{\Omega}$, in which all vertices except for N^ϵ many have radius $\mathfrak{R}$ tree neighborhood. Thus

$$\mathbb{P}_{\mathbf{S}}(\mathcal{B}_\mathfrak{R}(v, \mathcal{G}) \text{ is not a tree for some } v \in \mathcal{B}_\ell(\{b_\alpha, c_\alpha\}, \mathcal{G})) \leq N^{-1}N^\epsilon d(d-1)^\ell \leq N^{-1+3\epsilon/2}. \quad (8.8) \quad \{\mathbf{f}:\mathbf{small12}\}$$

The claim $\mathbb{P}_{\mathbf{S}}(\mathbf{F}(\mathcal{G})) \geq 1 - N^{-1+2\epsilon}$ follows from union bounding over all α using (8.7) and (8.8).

Under our assumption $I(\mathcal{F}, \mathcal{G}) = 1$, the radius $\mathfrak{R}$ neighborhood of o is a tree. Thus $\mu = d(d-1)^\ell$. Moreover, the neighborhoods $\mathcal{B}_{3\mathfrak{R}/2}(o, \mathcal{G})$, and $\mathcal{B}_\mathfrak{R}(\{b_\alpha, c_\alpha\}, \mathcal{G})$ for $\alpha \in \llbracket \mu \rrbracket$ are disjoint. It follows that $\text{dist}_{\mathcal{G}^{(\mathbb{T})}}(\{a_\alpha, b_\alpha, c_\alpha\}, \{a_\beta, b_\beta, c_\beta\}) > \mathfrak{R}/4$ for all $\alpha \neq \beta \in \llbracket \mu \rrbracket$, and the subgraph $\mathcal{B}_{\mathfrak{R}/4}(\{a_\alpha, b_\alpha, c_\alpha\}, \mathcal{G}^{(\mathbb{T})})$ after adding the edge $\{a_\alpha, b_\alpha\}$ is a tree for all $\alpha \in \llbracket \mu \rrbracket$. We conclude that $\mathbf{W}_{\mathbf{S}} = \llbracket \mu \rrbracket$.

Next we show that for any vertex $v \in \llbracket N \rrbracket$, the excess of $\mathcal{B}_\mathfrak{R}(v, \tilde{\mathcal{G}})$ is no bigger than that of $\mathcal{B}_\mathfrak{R}(v, \mathcal{G})$. Then it follows that $\tilde{\mathcal{G}} \in \bar{\Omega}$. If $\text{dist}(v, \{a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}) \geq \mathfrak{R}$, then $\mathcal{B}_\mathfrak{R}(v, \tilde{\mathcal{G}}) = \mathcal{B}_\mathfrak{R}(v, \mathcal{G})$, and the statement follows.

Otherwise either $v \in \mathcal{B}_{\mathfrak{R}}(\{a_\alpha\}_{\alpha \in [\mu]}, \mathcal{G}) \subset \mathcal{B}_{3\mathfrak{R}/2}(o, \mathcal{G})$ or $v \in \mathcal{B}_{\mathfrak{R}}(\{b_\alpha, c_\alpha\}, \mathcal{G})$ for some $\alpha \in [\mu]$. We will discuss the first case. The second case can be proven in the same way, so we omit its proof. If $v \in \mathcal{B}_\ell(o, \mathcal{G})$, we denote $\min_{\alpha \in [\mu]} \text{dist}_{\mathcal{G}}(v, \{l_\alpha\}) = r \leq \mathfrak{R}$. Then $\mathcal{B}_{\mathfrak{R}}(v, \tilde{\mathcal{G}})$ is a subgraph of $\mathcal{B}_{\mathfrak{R}}(v, \mathcal{G}) \cup_{\alpha \in [\mu]} \mathcal{B}_{\mathfrak{R}-r-1}(c_\alpha, \mathcal{G})$ after removing $\{(b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$ and adding $\{(l_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$. By our construction of $\mathcal{F}(\mathcal{G})$, $\mathcal{B}_{\mathfrak{R}-r-1}(c_\alpha, \mathcal{G})$ are disjoint trees. We conclude that $\mathcal{B}_{\mathfrak{R}}(v, \tilde{\mathcal{G}})$ is a tree. If $v \notin \mathcal{B}_\ell(o, \mathcal{G})$, we denote $\min_{\alpha \in [\mu]} \text{dist}_{\mathcal{G}}(v, a_\alpha) = r \leq \mathfrak{R}$, then $\mathcal{B}_{\mathfrak{R}}(v, \tilde{\mathcal{G}})$ is a subgraph of $\mathcal{B}_{\mathfrak{R}}(v, \mathcal{G}) \cup_{\alpha \in [\mu]} \mathcal{B}_{\mathfrak{R}-r-1}(b_\alpha, \mathcal{G})$ after removing $\{(b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$ and adding $\{(a_\alpha, b_\alpha)\}_{\alpha \in [\mu]}$. Again by our construction of $\mathcal{F}(\mathcal{G})$, $\mathcal{B}_{\mathfrak{R}-r-1}(c_\alpha, \mathcal{G})$ are disjoint trees, we conclude the excess of $\mathcal{B}_{\mathfrak{R}}(v, \tilde{\mathcal{G}})$ is at most that of $\mathcal{B}_{\mathfrak{R}}(v, \mathcal{G})$.

The claim $I(\mathcal{F}^+, \mathcal{G}) = 1$ follows from the construction of $\mathcal{F}(\mathcal{G})$. It also follows from the above discussion that $\mathcal{B}_{\mathfrak{R}}(v, \tilde{\mathcal{G}})$ is a tree for any $v \in \mathcal{B}_\ell(o, \mathcal{G})$. One can then check that $I(\mathcal{F}, \tilde{\mathcal{G}}) = 1$. This finishes the proof of the second statement in Lemma 8.2. $\square$

Number of “good” embeddings. From (8.4), each connected component of $\mathcal{F}$ is either an unused core edge or a radius- $(\ell + 1)$ ball corresponding to a used core edge. The following proposition states that the total number of embeddings where $I(\mathcal{F}, \mathcal{G}) = 1$ is approximately equal to that of choosing each connected component independently. {p:sumA}

Proposition 8.3. *Given a template $\mathcal{F} = (V, E)$ with core edges $\mathcal{C}$ and unused core edges $\mathcal{C}^\circ$ as in (8.4), as well as a d -regular graph $\mathcal{G} \in \bar{\Omega}$, we have*

$$\sum_{\mathbf{i}} I(\mathcal{F}, \mathcal{G}) = Z_{\mathcal{F}} \left(1 + O\left(\frac{1}{N^{1-2\epsilon}}\right) \right),$$

where

$$Z_{\mathcal{F}} := (Nd)^{|\mathcal{C}|} \left([(d-1)!]^{1+d+d(d-1)+\dots+d(d-1)^{\ell-1}} \right)^{|\mathcal{C} \setminus \mathcal{C}^\circ|}. \quad (8.9) \quad \{\mathbf{e:sumA2}\}$$

Here $|\mathcal{C}|$ is the number of core edges; and $|\mathcal{C} \setminus \mathcal{C}^\circ|$ is the number of used core edges. We remark that $Z_{\mathcal{F}}$ depends only on the template $\mathcal{F}$ but not $\mathcal{G}$.

Remark 8.4. *In the rest of this article, we have many expressions in the following form*

$$\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E}[I(\mathcal{F}, \mathcal{G}) \times (\dots)] \quad (8.10)$$

Thanks to Proposition 8.3, the above expression can be viewed as an average over all possible embedding of the template $\mathcal{F}$ into $\mathcal{G}$.

Proof. We notice that $|\mathcal{C}|$ is also the number of connected components of $\mathcal{F}$, and $|\mathcal{C} \setminus \mathcal{C}^\circ|$ is the number of connected components in $\mathcal{F}$ which are balls of radius $\ell + 1$.

We can prove (8.9) by induction on the number of connected components. If $\mathcal{F}$ consists of a single edge $\mathcal{F} = \{b, c\}$ which is an unused core edge, then

$$\sum_{\mathbf{i}} I(\mathcal{F}, \mathcal{G}) = \sum_{b,c} A_{bc} \prod_{v \in \mathcal{B}_\ell(c, \mathcal{G})} \mathbf{1}(\mathcal{B}_{\mathfrak{R}}(v, \mathcal{G}) \text{ is a tree}) = Nd \left(1 + O\left(\frac{1}{N^{1-3\epsilon/2}}\right) \right), \quad (8.11) \quad \{\mathbf{e:single_ed}\}$$

where we used the definition of $\bar{\Omega}$ from Definition 3.1. If $\mathcal{F}$ consists of a radius $(\ell + 1)$ -ball, corresponding to one used core edge, then we can also first sum over its core edge. The number of choices of this is the same as (8.11). ~~The single edge~~ The single edge over the remaining vertices. Each interior vertex of the radius- $(\ell + 1)$ ball contributes a factor $(d-1)!$, since there are $(d-1)!$ ways to embed its children vertices. We get

$$\sum_{\mathbf{i}} I(\mathcal{F}, \mathcal{G}) = Nd[(d-1)!]^{1+d+d(d-1)+\dots+d(d-1)^{\ell-1}} \left(1 + O\left(\frac{1}{N^{1-3\epsilon/2}}\right) \right). \quad (8.12) \quad \{\mathbf{e:ball}\}$$

If the statement holds for $\mathcal{F}$ with θ connected components, next we show it for $\mathcal{F}$ with $\theta + 1$ connected components. We can first sum over the indices corresponding to a connected component, fixing the other indices. If it is a single edge, we get a factor similar to (8.11); if it is a radius- $(\ell + 1)$ ball, we get a factor similar to (8.12). Next we can sum over the remaining θ connected components of $\mathcal{F}$, which gives (8.9). $\square$

8.1 Admissible functions

Definition 8.5 (Local Green's Function). *Given a forest $\mathcal{F} = (\mathbf{i}, E)$ embedded in $\mathcal{G}$, we introduce the local Green's function $L(z, \mathcal{F}, \mathcal{G})$: for $w, w' \in \mathbf{i}$,*

$$L_{ww'}(z, \mathcal{F}, \mathcal{G}) = P_{ww'}(\mathcal{F}, z, m_{\text{sc}}(z)). \quad (8.13)$$

Given the event $I(\mathcal{F}, \mathcal{G}) = 1$, $L_{ww'}$ is simply the Green's function of copies of d -regular trees (recall from Proposition 3.3): $L_{ww'} = 0$ if w, w' are disconnected in $\mathcal{F}$, otherwise,

$$L_{ww'} = m_d(z) \left(-\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{\text{dist}_{\mathcal{F}}(w, w')}. \quad (8.14)$$

We also denote the centered version of the Green's function as

$$G_{ww'}^{\circ}(z) = G_{ww'}(z) - L_{ww'}(z, \mathcal{F}, \mathcal{G}). \quad (8.15)$$

When the context is clear, we will simply write $G^{\circ}(z), L(z, \mathcal{F}, \mathcal{G})$ as G°, L for simplicity.

Remark 8.6. *At each step, we expand the forest $\mathcal{F} = (\mathbf{i}, E)$ to a new forest $\mathcal{F}^+ = (\mathbf{i}^+, E^+)$ by including local resampling data ($\mathcal{F} \subset \mathcal{F}^+$ are embedded subgraph of $\mathcal{G}$). Given the event $I(\mathcal{F}^+, \mathcal{G}) = 1$, the local Green's functions are compatible (both are given by the Green's function of copies of d -regular trees)*

$$L_{ww'}(z, \mathcal{F}, \mathcal{G}) = L_{ww'}(z, \mathcal{F}^+, \mathcal{G}), \quad w, w' \in \mathbf{i}.$$

Later, we need the local Green's function with one vertex removed: Let $(i, o) \in \mathcal{C} \setminus \mathcal{C}^{\circ}$, and recall $P^{(i)}$ from (3.5),

$$L_{ww'}^{(i)} := L_{ww'}^{(i)}(z, \mathcal{F}, \mathcal{G}) := P_{ww'}^{(i)}(\mathcal{F}, z, m_{\text{sc}}(z)). \quad (8.16)$$

Definition 8.7 (Admissible Function). *Consider a forest $\mathcal{F} = (\mathbf{i}, E)$ as defined in (8.4), with identifying edges $\mathcal{K}$, core edges $\mathcal{C}$, unused core edges $\mathcal{C}^{\circ}$. For any nonnegative integers $r \geq 0$, we denote the set of admissible functions $\text{Adm}(r, \mathcal{F}, \mathcal{G})$ where a function $\mathbf{R}_i \in \text{Adm}(r, \mathcal{F}, \mathcal{G})$ contains r factors of the form*

$$\{(G_{cc}^{(b)} - Q)\}_{(b,c) \in \mathcal{C}^{\circ}}, \quad \{G_{cc'}^{(bb')}, G_{bc'}^{(b')}, G_{bb'}\}_{(b,c) \neq (b',c') \in \mathcal{C}^{\circ}}, \quad \{G_{ss'}^{\circ}\}_{s,s' \in \mathcal{K}}, \quad (Q - m_{\text{sc}}(z)). \quad (8.17)$$

Remark 8.8. *At each step, we expand the forest $\mathcal{F}$ to a new forest $\mathcal{F}^+$ by including local resampling data. This change also affects the admissible set of functions, which now expands as follows:*

$$\text{Adm}(r, \mathcal{F}, \mathcal{G}) \subset \text{Adm}(r, \mathcal{F}^+, \mathcal{G}).$$

We now give the general ways of bounding the terms involved in the admissible functions (recall from Definition 8.7).

Proposition 8.9. *We take $z \in \mathbf{D}$. Then the following holds:*

1. *For any factor B in (8.17),*

$$\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) |B| \lesssim N^{-b}. \quad (8.18)$$

2. Let $R_i \in \text{Adm}(r, \mathbf{r}, \mathcal{F}, \mathcal{G})$ as in Definition 8.7. If R_i contains two terms of the form $\{G_{cc'}^{(bb')}, G_{bc'}^{(b')}, G_{bb'}\}$ with $(c, b) \neq (c', b') \in \mathcal{C}^\circ$

$$\frac{(d-1)^{3\ell r}}{Z_{\mathcal{F}}} \sum_i \mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) R_i = O\left(\frac{(d-1)^{3\ell r} N^o}{N^{(r-2)\mathfrak{b}}} \Phi\right). \quad (8.19) \quad \{\mathfrak{e}:\text{refined_b}\}$$

If we further assume $r \geq 3$, then the expectation of (8.19) is $\frac{(d-1)^{3\ell r}}{N^{(r-2)\mathfrak{b}}} \mathbb{E}[\Psi_p]$; \{\mathfrak{e}:\text{refined_b}\}

The claim (8.18) follows from the definition (3.21) of the set Ω and can be proved in the same way as Lemma 3.9. Thus, we omit its proof. For (8.19), we show the case that R_i contains $(G_{cc'}^{(bb')})^2$. The other cases can be proven in the same way so we omit. We can first bound the other factors of R_i using (8.18), as $N^{-(r-2)\mathfrak{b}}$. Then (8.19) reduces to

$$\frac{(d-1)^{3\ell r}}{Z_{\mathcal{F}}} \sum_i \mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) |R_i| \lesssim \frac{(d-1)^{3\ell r}}{N^{(r-2)\mathfrak{b}} Z_{\mathcal{F}}} \sum_i \mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) |G_{cc'}^{(bb')}|^2 \lesssim \frac{(d-1)^{3\ell r} \Phi}{N^{(r-2)\mathfrak{b}} Z_{\mathcal{F}}}, \quad (8.20)$$

which follows from the same argument as (7.51).

9 Switching using the Schur complement formula \{\text{sec:schurlem}\}

In this section we will use the Schur complement formula to study the Green's function after local resampling. We recall the local resampling and related notation from Section 5. We also introduce the following S-Product term. \{\mathfrak{d}:\text{S-product}\}

Definition 9.1 (S-Product term). Fix $r \geq 0$, we define R_r to be a S-product term of order r (where “S” indicates that these terms arise from expansions using the Schur complement formula) if it is a product of r factors in the following forms:

$$(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q), \quad G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}, \quad (Q - m_{\text{sc}}(z)), \alpha \neq \beta \in \llbracket \mu \rrbracket.$$

In the following proposition, we derive an expansion for factors that are Green's function entries with both indices in $\{i, o\}$. They can be proven in exactly the same way as Lemma 7.4 using the Schur complement formula, so we omit their proofs. \{\text{lem:diaglem}\}

Proposition 9.2. We assume that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$ (recall from (8.6)), and define the index set $A_i := \{\alpha \in \llbracket \mu \rrbracket : \text{dist}_{\mathcal{T}}(i, l_\alpha) = \ell + 1\}$ (see Figure 5). Then the following holds

1. $\tilde{G}_{oo}^{(i)} - Y$ can be rewritten as a weighted sum

$$\tilde{G}_{oo}^{(i)} - Y = \frac{m_{\text{sc}}(z)^{2\ell+2}}{(d-1)^{\ell+1}} \sum_{\alpha \in A_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) + \frac{m_{\text{sc}}(z)^{2\ell+2}}{(d-1)^{\ell+1}} \sum_{\alpha \neq \beta \in A_i} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{U} + \mathcal{E}. \quad (9.1) \quad \{\mathfrak{e}:\text{G-Y}\}$$

where $\mathcal{U}$ is an $O(1)$ -weighted sum of terms in the form $(d-1)^{3(r-1)\ell} R_r$, where $r \geq 2$, and R_r is an S-product term (see Definition 9.1) which contains $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$ or $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$; and the error $\mathcal{E}$ is given by

$$\begin{aligned} \mathcal{E} &= \frac{\mathfrak{c}}{(d-1)^{\ell+1}} \sum_{\alpha, \beta \in A_i} (\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}) + O\left(\frac{1}{N^{\mathfrak{b}/2}} \sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + \frac{1}{N^2}\right) \\ &= O\left(\sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + \frac{1}{N^2}\right). \end{aligned} \quad (9.2) \quad \{\mathfrak{e}:\text{defCE_cop}\}$$

2. For $s, s' \in \{i, o\}$, $\tilde{G}_{ss'}^\circ = (\tilde{G} - \tilde{L})_{ss'}$ can be rewritten as a weighted sum,

$$\begin{aligned} \tilde{G}_{ss'}^\circ &= \frac{1}{(d-1)^\ell} \sum_{\alpha \in \llbracket \mu \rrbracket} \mathbf{c}_1(\mathbf{1}(\alpha \in \mathbf{A}_i))(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) \\ &\quad + \frac{1}{(d-1)^\ell} \sum_{\alpha \neq \beta \in \llbracket \mu \rrbracket} \mathbf{c}_2(\mathbf{1}(\alpha \in \mathbf{A}_i), \mathbf{1}(\beta \in \mathbf{A}_i))G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{U} + \mathcal{E}, \end{aligned} \quad (9.3) \quad \{\mathbf{e}:\mathbf{G}\text{-Ydouble}\}$$

where $|\mathbf{c}_1(\cdot)|, |\mathbf{c}_2(\cdot, \cdot)| \lesssim 1$; $\mathcal{U}$ is an $O(1)$ -weighted sum of terms of the form $(Q - m_{\text{sc}}(z))$ or $(d-1)^{3(r-1)\ell} R_r$, where $r \geq 2$, and R_r is an S -product term (see Definition 9.1); and the error $\mathcal{E}$ is bounded by

$$|\mathcal{E}| \lesssim \sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + \frac{1}{N^2}.$$

In the following proposition, we derive a similar expansion for factors that are Green's function entries with at most one index in $\{i, o\}$. \{lem:offdiags\}

Proposition 9.3. We assume that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$ (recall from (8.6)), ~~\{end-of-diagonal\}~~ the index set $\mathbf{A}_i := \{\alpha \in \llbracket \mu \rrbracket : \text{dist}_{\mathcal{T}}(i, l_\alpha) = \ell + 1\}$ (see Figure 5). Then for any unused core edges $(b, c) \neq (b', c') \in \mathcal{C}^\circ \setminus \{(i, o)\}$, the following holds:

1. $\tilde{G}_{oc}^{(ib)}$ and $\tilde{G}_{ic}^{(b)}$ can be rewritten as a weighted sum

$$\frac{1}{(d-1)^{\ell/2}} \sum_{\alpha \in \llbracket \mu \rrbracket} \mathbf{c}_1(\mathbf{1}(\alpha \in \mathbf{A}_i))G_{c_\alpha c}^{(b_\alpha b)} + \mathcal{U} + \mathcal{E}, \quad (9.4) \quad \{\mathbf{e}:\mathbf{tGoc_exp}\}$$

where $|\mathbf{c}_1(\cdot)| \lesssim 1$; $\mathcal{U}$ is an $O(1)$ -weighted sum of terms of the form $(d-1)^{3r\ell} R_r G_{c_\alpha c}^{(b_\alpha b)}$, for R_r an S -product term (see Definition 9.1) with $r \geq 1$, and the error $\mathcal{E}$ is bounded by

$$|\mathcal{E}| \lesssim \sum_{\alpha \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c}^{(\mathbb{T}b)} - G_{c_\alpha c}^{(b_\alpha b)}| + \sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + N^{-2}. \quad (9.5) \quad \{\mathbf{e}:\mathbf{tGoc_cE}\}$$

2. $\tilde{G}_{ib}$ and $\tilde{G}_{ob}^{(i)}$ can be rewritten as a weighted sum

$$\frac{1}{(d-1)^{\ell/2}} \sum_{\alpha \in \llbracket \mu \rrbracket} \mathbf{c}_1(\mathbf{1}(\alpha \in \mathbf{A}_i))G_{c_\alpha b}^{(b_\alpha)} + \mathcal{U} + \mathcal{E},$$

where $|\mathbf{c}_1(\cdot)| \lesssim 1$; $\mathcal{U}$ is an $O(1)$ -weighted sum of terms of the form $(d-1)^{3r\ell} R_r G_{c_\alpha b}^{(b_\alpha)}$, where R_r is an S -product term (see Definition 9.1) with $r \geq 1$; and the error $\mathcal{E}$ is bounded by

$$|\mathcal{E}| \lesssim \sum_{\alpha \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha b}^{(\mathbb{T})} - G_{c_\alpha b}^{(b_\alpha)}| + \sum_{\alpha, \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| + N^{-2}.$$

3. $\tilde{G}_{cc}^{(b)}, \tilde{G}_{cc'}^{(bb')}, \tilde{G}_{bc'}^{(b')}$ and $\tilde{G}_{bb'}$ can be rewritten as

$$\begin{aligned} \tilde{G}_{cc}^{(b)} &= G_{cc}^{(b)} + \mathcal{E}, \quad |\mathcal{E}| \lesssim |\tilde{G}_{cc}^{(\mathbb{T}b)} - G_{cc}^{(b)}| + (d-1)^\ell \sum_{\alpha \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c}^{(\mathbb{T}b)}|^2 + (d-1)^{2\ell} N^{2\alpha} t\Phi, \\ \tilde{G}_{cc'}^{(bb')} &= G_{cc'}^{(bb')} + \mathcal{E}, \quad |\mathcal{E}| \lesssim |\tilde{G}_{cc'}^{(\mathbb{T}bb')} - G_{cc'}^{(bb')}| + (d-1)^\ell \sum_{\alpha \in \llbracket \mu \rrbracket} (|\tilde{G}_{c_\alpha c}^{(\mathbb{T}bb')}|^2 + |\tilde{G}_{c' c_\alpha}^{(\mathbb{T}bb')}|^2), \\ \tilde{G}_{bc'}^{(b')} &= G_{bc'}^{(b')} + \mathcal{E}, \quad |\mathcal{E}| \lesssim |\tilde{G}_{bc'}^{(\mathbb{T}b')} - G_{bc'}^{(b')}| + (d-1)^\ell \sum_{\alpha \in \llbracket \mu \rrbracket} (|\tilde{G}_{bc_\alpha}^{(\mathbb{T}b')}|^2 + |\tilde{G}_{c' c_\alpha}^{(\mathbb{T}b')}|^2), \\ \tilde{G}_{bb'} &= G_{bb'} + \mathcal{E}, \quad |\mathcal{E}| \lesssim |\tilde{G}_{bb'}^{(\mathbb{T})} - G_{bb'}| + (d-1)^\ell \sum_{\alpha \in \llbracket \mu \rrbracket} (|\tilde{G}_{bc_\alpha}^{(\mathbb{T})}|^2 + |\tilde{G}_{b' c_\alpha}^{(\mathbb{T})}|^2). \end{aligned}$$

10 Switching using the Woodbury formula

{sec:analysis}

In this section, we introduce a novel expansion based on the Woodbury formula (2.2). If the forest $\mathcal{F}$ of this section, we assume that $I(\mathcal{F}^+, \mathcal{G}) = 1$ (recall from (8.6)). Then the switching edges $(b_\alpha, c_\alpha)_{\alpha \in [\mu]}$ are far away from each other, and have large tree neighborhood. Thanks to Lemma 8.2, $I(\mathcal{F}^+, \mathcal{G}) = 1$ holds with high probability provided $I(\mathcal{F}, \mathcal{G}) = 1$.

We compare the normalized adjacency matrix of the switched graph to that of the original graph, $\tilde{H} - H$. We denote the adjacency matrices of our switching as

$$\tilde{H} - H = - \sum_{\alpha \in [\mu]} \xi_\alpha, \quad \xi_\alpha := \frac{1}{\sqrt{d-1}} (\Delta_{l_\alpha a_\alpha} + \Delta_{b_\alpha c_\alpha} - \Delta_{l_\alpha c_\alpha} - \Delta_{a_\alpha b_\alpha}).$$

We denote the rank of this difference as $r = O((d-1)^\ell)$, and rewrite

$$\tilde{H} - H = UV^\top,$$

where U, V are $N \times r$ matrices, and their nonzero rows correspond to the vertices $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}$. Then, the Woodbury formula (2.2) gives us

$$\tilde{G} - G = (H - z + UV^\top)^{-1} - (H - z)^{-1} = -GU(\mathbb{I} + V^\top GU)^{-1}V^\top G. \quad (10.1) \quad \{\mathbf{e: tGG}\}$$

We recall $\mathcal{F}^+$ as in (8.3), and denote by $\tilde{\mathcal{F}}^+$ the switched version of it

$$\mathcal{F}^+ := \mathcal{F} \cup \mathcal{B}_\ell(o, \mathcal{G}) \cup \bigcup_{\alpha=1}^{\mu} \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}, \quad \tilde{\mathcal{F}}^+ := \mathcal{F} \cup \mathcal{B}_\ell(o, \mathcal{G}) \cup \bigcup_{\alpha=1}^{\mu} \{(l_\alpha, c_\alpha), (a_\alpha, b_\alpha)\}.$$

We view $\mathcal{F}^+, \tilde{\mathcal{F}}^+$ as subgraphs of $\mathcal{G}, \tilde{\mathcal{G}}$ respectively. We will analyze (10.1) using local Green's functions

$$L := P(\mathcal{F}^+, z, m_{\text{sc}}(z)), \quad \tilde{L} := P(\tilde{\mathcal{F}}^+, z, m_{\text{sc}}(z)), \quad (10.2) \quad \{\mathbf{e: defPtP}\}$$

as was defined in Definition 3.4. We remark that condition on $I(\mathcal{F}^+, \mathcal{G}) = 1$, both L and $\tilde{L}$ are simply the Green's function of copies of d -regular trees.

Notice that when restricted to the vertex set of $\mathcal{F}^+$ (which contains the vertices $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}$),

$$\tilde{L}^{-1} - L^{-1} = \tilde{H} - H = - \sum_{\alpha \in [\mu]} \xi_\alpha = UV^\top. \quad (10.3) \quad \{\mathbf{e: inverseL}\}$$

We can use the Woodbury formula on $L, \tilde{L}$ as well, giving

$$\tilde{L} - L = -LU(\mathbb{I} + V^\top LU)^{-1}V^\top L. \quad (10.4) \quad \{\mathbf{e: tP-P}\}$$

A crucial observation is that the quantity $-U(\mathbb{I} + V^\top LU)^{-1}V^\top$ in (10.1) and (10.4) take very simple form. {1: defF2}

Lemma 10.1. *We introduce the following matrix F , which is nonzero on the vertex set $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}$,*

$$F := \sum_{\alpha \in [\mu]} \xi_\alpha + \sum_{\alpha, \beta \in [\mu]} \xi_\alpha \tilde{L} \xi_\beta. \quad (10.5) \quad \{\mathbf{e: defF}\}$$

Then

$$F = -U(\mathbb{I} + V^\top LU)^{-1}V^\top. \quad (10.6) \quad \{\mathbf{e: defF2g}\}$$

Proof of Lemma 10.1. The nonzero rows of U, V are parametrized by $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}$. By rearranging the above expression (10.4) (we view F as restricted on the vertex set of $\mathcal{F}^+$), we get

$$L^{-1}\tilde{L}L^{-1} - L^{-1} = -U(\mathbb{I} + V^\top LU)^{-1}V^\top. \quad (10.7) \quad \{\mathbf{e:Fz}\}$$

We can reorganize (10.7) as

$$\begin{aligned} -U(\mathbb{I} + V^\top LU)^{-1}V^\top &= L^{-1}\tilde{L}L^{-1} - L^{-1} = L^{-1}\tilde{L}\tilde{L}^{-1} + L^{-1}\tilde{L}(L^{-1} - \tilde{L}^{-1}) - L^{-1} \\ &= L^{-1}\tilde{L}(L^{-1} - \tilde{L}^{-1}) = (L^{-1} - \tilde{L}^{-1})\tilde{L}(L^{-1} - \tilde{L}^{-1}) + \tilde{L}^{-1}\tilde{L}(L^{-1} - \tilde{L}^{-1}) \\ &= (L^{-1} - \tilde{L}^{-1}) + (L^{-1} - \tilde{L}^{-1})\tilde{L}(L^{-1} - \tilde{L}^{-1}) = \sum_{\alpha \in [\mu]} \xi_\alpha + \sum_{\alpha, \beta \in [\mu]} \xi_\alpha \tilde{L} \xi_\beta = F, \end{aligned} \quad (10.8) \quad \{\mathbf{e: defF2}\}$$

where in the last statement we used (10.3). $\square$

Our next lemma attempts to expand $\tilde{G} - G$ in terms of $\tilde{L} - L$.

Lemma 10.2. *We assume that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$ (recall from (8.6)), and recall $G^\circ = (G - L)$. Then with overwhelmingly high probability over Z , we have:*

$$\tilde{G} - G = \sum_{k \geq 0} GF(G^\circ F)^k G, \quad (10.9) \quad \{\mathbf{e:tG-Gdiff}\}$$

Proof. Thanks to (3.21), we find for $x, y \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}$. We can then expand (10.1) using the resolvent identity (2.1) and (10.6) to conclude that

$$\begin{aligned} \tilde{G} - G &= -GU(\mathbb{I} + V^\top GU)^{-1}V^\top G = -GU(\mathbb{I} + V^\top LU + V^\top G^\circ U)^{-1}V^\top G \\ &= -GU \left((\mathbb{I} + V^\top LU)^{-1} \sum_{k \geq 0} (-1)^k (V^\top G^\circ U (\mathbb{I} + V^\top LU)^{-1})^k \right) V^\top G \\ &= \sum_{k \geq 0} (-1)^{k+1} GU(\mathbb{I} + V^\top LU)^{-1} (V^\top G^\circ U (\mathbb{I} + V^\top LU)^{-1})^k V^\top G \\ &= \sum_{k \geq 0} GF(G^\circ F)^k G. \end{aligned}$$

This gives (10.9). $\square$

Proposition 10.3. *We assume that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$ (recall from (8.6)). Then for $w, w' \in [N]$, we have*

$$|(\tilde{G} - G)_{ww'}| \leq (d-1)^\ell \sum_{x \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}} (|G_{wx}|^2 + |G_{w'x}|^2). \quad (10.10) \quad \{\mathbf{e:Gsw_exp}\}$$

As a consequence, we have the following bounds

$$|\tilde{Q} - Q|, |\tilde{m} - m| \lesssim (d-1)^{6\ell} N^\circ \Phi. \quad (10.11) \quad \{\mathbf{e:Qtmtbound}\}$$

We start with the following estimates, which will be used later to prove Proposition 10.3.

Claim 10.4. *The matrix F from (10.5) has nonzero entries only on the vertices $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}$, and*

$$\sum_{s, s' \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}} |F_{ss'}| \lesssim \ell(d-1)^\ell. \quad (10.12) \quad \{\mathbf{e:Fbound}\}$$

Proof of Claim 10.4. It is easy to see from the expression (10.5) that $F_{\alpha\alpha}$ has nonzero entries only on the vertices $\{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}$. We remark that condition on $I(\mathcal{F}^+, \mathcal{G}) = 1$, both L and $\tilde{L}$ are simply the Green's function of copies of d -regular trees. Hence, the estimate (3.1) gives

$$|\tilde{L}_{J_\alpha J'_\alpha}| \lesssim (d-1)^{-\text{dist}_\mathcal{T}(l_\alpha, l_{\alpha'})/2}, \text{ for } J, J' \in \{l, a, b, c\}, \quad \alpha, \alpha' \in \llbracket \mu \rrbracket. \quad (10.13) \quad \{\mathbf{e:Piolha}\}$$

By plugging (10.13) into (10.11), we get

$$|F_{J_\alpha J'_\alpha}| \lesssim (d-1)^{-\text{dist}_\mathcal{T}(l_\alpha, l_{\alpha'})/2}, \text{ for } J, J' \in \{l, a, b, c\}, \quad \alpha, \alpha' \in \llbracket \mu \rrbracket, \\ \sum_{s, s' \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}} |F_{ss'}| \lesssim \sum_{\alpha, \alpha' \in \llbracket \mu \rrbracket} (d-1)^{-\text{dist}_\mathcal{T}(l_\alpha, l_{\alpha'})/2} \lesssim \ell(d-1)^\ell,$$

where in the last inequality we used that $\{\alpha' \in \llbracket \mu \rrbracket : \text{dist}_\mathcal{T}(l_\alpha, l_{\alpha'}) = 2r\} = O((d-1)^r)$ for $0 \leq r \leq \ell$. $\square$

Proof of Proposition 10.3. For any $x, y \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}$, we have

$$|(F(G^\circ F)^k)_{xy}| \leq N^{-k\mathbf{b}} \sum_{s_1, s_2} |F_{s_1 s_2}| \sum_{s_3, s_4} |F_{s_3 s_4}| \cdots \sum_{s_{2k-1}, s_{2k}} |F_{s_{2k-1} s_{2k}}| \leq N^{-k\mathbf{b}} \ell^k (d-1)^{k\ell} \quad (10.14) \quad \{\mathbf{e:sumF}\}$$

where in the first statement we used $|G_{xy}^\circ| \leq N^{-\mathbf{b}}$ uniformly for $x, y \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}$ from (3.21); in the second statement we used (10.12).

The claim (10.10) follows from (10.9),

$$|(\tilde{G} - G)_{ww'}| = \left| \sum_{k \geq 0} (GF(G^\circ F)^k G)_{ww'} \right| \leq \sum_{x, y \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}} \sum_{k \geq 0} (\ell N^{-\mathbf{b}} (d-1)^\ell)^k |G_{wx}| |G_{yw'}| \\ \leq (d-1)^\ell \sum_{x, y \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}} |G_{wx}| |G_{yw'}| \quad (10.15) \quad \{\mathbf{e:tG-Gdiff}\} \\ \leq (d-1)^\ell \sum_{x \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in \llbracket \mu \rrbracket}} (|G_{wx}|^2 + |G_{w'x}|^2).$$

where in the first statement we used (10.9); in the second statement we used (10.14); in the third statement we sum the geometry series; in the last statement we used Cauchy-Schwartz inequality.

Next we prove (10.11) for $|\tilde{Q} - Q|$, the statement for $|\tilde{m} - m|$ follows from (5.8). The difference $\tilde{Q} - Q$ can be rewritten as

$$\frac{1}{Nd} \sum_{\{u, v\} \notin \{\{l_\alpha, a_\alpha\}, \{b_\alpha, c_\alpha\}\}_{\alpha \in \llbracket \mu \rrbracket}} A_{uv} (\tilde{G}_{vv}^{(u)} - G_{vv}^{(u)}) + \frac{1}{Nd} \sum_{\{u, v\} \in \{\{l_\alpha, c_\alpha\}, \{a_\alpha, b_\alpha\}\}_{\alpha \in \llbracket \mu \rrbracket}} (\tilde{G}_{vv}^{(u)} - G_{vv}^{(u)}) \\ + \frac{1}{Nd} \sum_{\{u, v\} \in \{\{l_\alpha, c_\alpha\}, \{a_\alpha, b_\alpha\}\}_{\alpha \in \llbracket \mu \rrbracket}} G_{vv}^{(u)} - \frac{1}{Nd} \sum_{\{u, v\} \in \{\{l_\alpha, a_\alpha\}, \{b_\alpha, c_\alpha\}\}_{\alpha \in \llbracket \mu \rrbracket}} G_{vv}^{(u)}. \quad (10.16) \quad \{\mathbf{e:tQ-Qdiff}\}$$

For $G_{vv}^{(u)}, \tilde{G}_{vv}^{(u)}$ in (10.16), we can rewrite them using the Schur complement formula (2.5),

$$G_{vv}^{(u)} = G_{vv} - \frac{G_{uv}^2}{G_{uu}}, \quad \tilde{G}_{vv}^{(u)} = \tilde{G}_{vv} - \frac{\tilde{G}_{uv}^2}{\tilde{G}_{uu}}. \quad (10.17) \quad \{\mathbf{e:schurexp}\}$$

For the difference $\tilde{G}_{vv}^{(u)} - G_{vv}^{(u)}$, using (10.17) and (3.21), we can bound it as

$$|\tilde{G}_{vv}^{(u)} - G_{vv}^{(u)}| \leq |\tilde{G}_{vv} - G_{vv}| + |\tilde{G}_{uv} - G_{uv}| + |\tilde{G}_{uu} - G_{uu}|. \quad (10.18) \quad \{\mathbf{e:Guuvdiff}\}$$

By plugging (10.18) into (10.16), we conclude that

$$|\tilde{Q} - Q| \lesssim \sum_{u \sim v} (|\tilde{G}_{vv} - G_{vv}| + |\tilde{G}_{uv} - G_{uv}| + |\tilde{G}_{uu} - G_{uu}|) + \frac{(d-1)^\ell}{N} \quad (10.19)$$

$$\lesssim (d-1)^\ell \sum_{u \sim v} \sum_{x \in \{l_\alpha, a_\alpha, b_\alpha, c_\alpha\}_{\alpha \in [\mu]}} (|G_{ux}|^2 + |G_{vx}|^2) \lesssim (d-1)^{3\ell} N^\circ \Phi, \quad (10.20)$$

where in the last inequality, we used the Ward identity bound. $\square$

11 Proof for the self-consistent equation

In this section we prove (4.23) in Theorem 4.5. As discussed in Section 8, at each iteration, we estimate (8.1) by performing a local resampling around (i, o) . We will show that the expectation breaks down into an $O(1)$ -weighted sum of terms in the same form. We begin with a weighted version of (8.1), as presented on the left-hand side of (11.1) in the following proposition. Here, the additional factor $(d-1)^{3r\ell}$ depends on the admissible function R_i . The reader can interpret this as follows: each term in (8.17) (there are r such terms) is associated with a factor $(d-1)^{3\ell}$. Thanks to Equation (8.18), even with these factors, the size of the terms remains small, i.e. bounded by $(d-1)^{3r\ell} N^{-br} \ll 1$. These factors are introduced to ensure that all the expansions in this section are $O(1)$ -weighted sums of terms, as defined in Definition 7.2. Specifically, combinatorial factors are absorbed into $(d-1)^{3r\ell}$.

The proposition below expresses the expectation of Green's functions of the graph $\mathcal{G}$ in terms of the quantities of the new graph $\tilde{\mathcal{G}}$ after local resampling.

Proposition 11.1. *Consider a forest $\mathcal{F} = (\mathbf{i}, E)$ as in (8.4) and a function $(G_{oo}^{(i)} - Y)R_i$ with $R_i \in \text{Adm}(r, \mathcal{G}, \mathcal{F})$ and $r \geq 1$. We perform a local resampling around $(i, o) \in \mathcal{F}$ using the resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$, denoting the new graph as $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$, with its corresponding Green's function $\tilde{G}$. Then*

$$\begin{aligned} & \frac{(d-1)^{3r\ell}}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{oo}^{(i)} - Y) R_i \right] \\ &= \frac{(d-1)^{3r\ell}}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \tilde{R}_i \right] + O(N^{-b/2} \mathbb{E}[\Psi]). \end{aligned} \quad (11.1)$$

Here, $\tilde{R}_i$ is obtained by computing R_i for the graph $\tilde{\mathcal{G}}$.

If we temporarily ignore the indicator and the averaging over embeddings, the above proposition reduces to the symmetry

$$\mathbb{E} \left[(G_{oo}^{(i)} - Y) R_i \right] = \mathbb{E} \left[(\tilde{G}_{oo}^{(i)} - \tilde{Y}) \tilde{R}_i \right] = \mathbb{E} \left[(\tilde{G}_{oo}^{(i)} - Y) \tilde{R}_i \right] - \mathbb{E} \left[(\tilde{Y} - Y) \tilde{R}_i \right].$$

The indicator $I(\mathcal{F}, \mathcal{G})$ just restrict attention to “good” placements where we have good estimates for the Green's function. In this regime we have $|\tilde{R}_i| \lesssim N^{-b}$ and

$$|\tilde{Y} - Y| = |Y_\ell(\tilde{Q}, z) - Y_\ell(Q, z)| \lesssim |\ell(\tilde{Q} - Q)| \lesssim \ell(d-1)^{6\ell} N^\circ \Phi \quad (11.2)$$

where we used (3.9) and (10.11). So the last term above is negligible. We omit the proof of Proposition 11.1.

The right-hand side of (11.1) involves the Green's function of the switched graph $\tilde{\mathcal{G}}$. The following two propositions help evaluate them, and express them as $O(1)$ -weighted sums of terms involving only the Green's function of the original graph $\mathcal{G}$, with negligible error. More importantly, these terms match the structure of the left-hand side of (11.1).

{p:general}

Proposition 11.2. *Given a forest $\mathcal{F} = (\mathbf{i}, E)$ and a function $(G_{oo}^{(i)} - Y)R_{\mathbf{i}}$ with $R_{\mathbf{i}} \in \text{Adm}(r, \mathcal{F}, \mathcal{G})$. We construct $\mathcal{F}^+ = (\mathbf{i}^+, E^+)$ (as given by (8.5)) by performing a local resampling around $(i, o) \in \mathcal{F}$ with resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$, and denote $\tilde{\mathcal{G}} = T_{\mathbf{S}}(\mathcal{G})$.*

1. Let $r \geq 2$ and take $R_{\mathbf{i}} \in \text{Adm}(r, \mathcal{F}, \mathcal{G})$. Then, up to an error of size $O(N^{-b/4}\mathbb{E}[\Psi])$,

$$\frac{(d-1)^{3r\ell}}{(d-1)^{q\ell/2}Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \tilde{R}_{\mathbf{i}} \right] \quad (11.3) \quad \{\mathbf{e}:\text{higher_ca}\}$$

can be rewritten as an $O(1)$ -weighted sum of terms in the following form

$$\frac{(d-1)^{3r^+\ell}}{(d-1)^{q^+\ell/2}Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) R_{\mathbf{i}^+}], \quad (11.4) \quad \{\mathbf{e}:\text{case3_cop}\}$$

where $R_{\mathbf{i}^+} \in \text{Adm}(r^+, \mathcal{F}^+, \mathcal{G})$, where either $q^+ \geq q+1$, $r^+ \geq r$; or $q^+ \geq q$, $r^+ \geq r+1$.

2. Let $R_{\mathbf{i}} = (G_{oo}^{(i)} - Q)$. Then, up to an error of size $O((d-1)^{-q\ell/2}N^o\mathbb{E}[\Psi])$,

$$\frac{1}{(d-1)^{q\ell/2}Z_{\mathcal{F}^+}} \sum_{\mathbf{i}} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \tilde{R}_{\mathbf{i}}] \quad (11.5) \quad \{\mathbf{e}:\text{higher_ca}\}$$

can be rewritten as an $O(1)$ -weighted sum of terms in the form of (11.4) with $R_{\mathbf{i}^+} \in \text{Adm}(r^+, \mathcal{F}^+, \mathcal{G})$, where $r^+ \geq 2$ and $q^+ \geq 0$, or in the following form

$$\frac{1}{(d-1)^{q^+\ell/2}Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)], \quad (11.6) \quad \{\mathbf{e}:\text{higher_ca}\}$$

where $q^+ \geq q+1$.

The claim (4.23) follows from iterating Proposition 11.1 and Proposition 11.2.

Proof of (4.23). We recall the sequence of forests $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots$ from Section 8. They encode all edges involved in local resamplings. By Claim 7.1 and Proposition 7.3, up to an error $(d-1)^{2\ell}\mathbb{E}[\Psi]$, $\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega)(Q - Y)]$ is an $O(1)$ -weighted sum of terms of the following two forms

$$\frac{1}{(d-1)^{q\ell/2}Z_{\mathcal{F}_1}} \sum_{\mathbf{i}} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}_1, \mathcal{G}) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)], \quad (11.7) \quad \{\mathbf{e}:\text{st1}\}$$

where $q \geq 0$; or

$$\frac{(d-1)^{3r\ell}}{(d-1)^{q\ell/2}Z_{\mathcal{F}_1}} \sum_{\mathbf{i}} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}_1, \mathcal{G}) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) R_{\mathbf{i}_1}], \quad (11.8) \quad \{\mathbf{e}:\text{st2}\}$$

where $q \geq 0$ and $r \geq 2$, and the function $R_{\mathbf{i}_1} \in \text{Adm}(r, \mathcal{F}_1, \mathcal{G})$.

The above expression (11.7) along with the form of Proposition 11.1, allowing us to apply Proposition 11.1 and Proposition 11.2 for the iteration process. The result in Proposition 11.2 essentially states that, after further expansion, (11.7) either maintain the same form with an additional factor of $(d-1)^{-\ell/2}$, or they transform into (11.8). Similarly, after expansion, (11.8) remains in the same form, either with an additional $(d-1)^{-\ell/2}$ factor, or an extra term in the form of (8.17), which is bounded by $N^{-b} \ll (d-1)^{-\ell/2}$. Therefore, after finitely many steps, namely $O(4\log_{d-1}(N)/\ell)$, all terms are bounded by $O(N^{-2}) = O(\mathbb{E}[\Psi]/N)$. Meanwhile the errors from Proposition 11.1 and Proposition 11.2 are all bounded by $O((d-1)^{2\ell}\mathbb{E}[\Psi])$. This gives (4.23). $\square$

11.1 Proof of Proposition 11.2

{s:general}

We denote $R_i \in \text{Adm}(r, \mathcal{F}, \mathcal{G})$ in Proposition 11.2 (as in Definition 8.7), which contains r factors in the form of (8.17).

In the following we prove Proposition 11.2 assuming $\mathfrak{q} = 0$. The case with $\mathfrak{q} \geq 1$ follows from simply multiplying $(d-1)^{-\mathfrak{q}\ell/2}$. In order to use our various propositions from Section ?? and Section 10 that allow us to reduce $(\tilde{G}_{oo}^{(i)} - Y)\tilde{R}_i$ to terms of the unswitched graph $\mathcal{G}$, we need to classify the factors of R_i based on their dependence on i, o . Let

$$B_0 = (G_{oo}^{(i)} - Y), \quad R_i = \prod_{j=1}^r B_j, \quad (11.9) \quad \{\mathfrak{e}:\text{introB_j}\}$$

where B_j is of one of the following forms

$$\begin{aligned} &\{G_{cc}^{(b)} - Q\}_{(b,c) \in \mathcal{C}^o}, \quad \{G_{oc}^{(ib)}, G_{ic}^{(b)}, G_{ib}, G_{ob}^{(i)}\}_{(i,o) \neq (b,c) \in \mathcal{C}^o}, \\ &\{G_{cc'}^{(bb')}, G_{bc'}^{(b')}, G_{bb'}\}_{(i,o) \neq (b,c), (b',c') \in \mathcal{C}^o}, \quad (Q - m_{\text{sc}}(z)). \end{aligned} \quad (11.10) \quad \{\mathfrak{e}:\text{Blist}\}$$

We can write (11.3) (with Blist) as

$$\begin{aligned} &\frac{(d-1)^{3r\ell}}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \tilde{R}_i \right] \\ &= \frac{(d-1)^{3r\ell}}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{oo}^{(i)} - Y) \prod_{j=1}^r \tilde{B}_j \right]. \end{aligned} \quad (11.11) \quad \{\mathfrak{e}:\text{sreplace0}\}$$

In the following we discuss the terms B_j as in (11.9) after ~~replacab~~ resampling. In Item 1-Item 5 below, for any $h \geq 0$, we denote R_h a product of h factors of the form

$$(G_{cc}^{(b)} - Q), \quad \{G_{cc'}^{(bb')}, G_{bc'}^{(b')}, G_{bb'}\}, \quad (Q - m_{\text{sc}}(z)), \quad (11.12) \quad \{\mathfrak{e}:\text{list_fact}\}$$

where $(b, c) \neq (b', c') \in \mathcal{K}^+$. They are terms from Proposition 9.2 and Proposition 9.3. We refer to Figure 7 for a picture illustration. {i1}

1. By Proposition 9.2, $\tilde{G}_{oo}^{(i)} - Y = \hat{B}_0 + \mathcal{E}_0$, where $\hat{B}_0$ is given by

$$\hat{B}_0 = \frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1}} \sum_{\alpha \in A_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) + \frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1}} \sum_{\alpha \neq \beta \in A_i} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{U}_0. \quad (11.13) \quad \{\mathfrak{e}:\text{fcase1W0}\}$$

Here, $\mathcal{U}_0$ is an $O(1)$ -weighted sum of terms of the form $(d-1)^{3(h-1)\ell} R_h$ with $h \geq 2$, and each term contains at least one factor of the form $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$ or $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$; $\mathcal{E}_0$ is $\mathcal{E}$ in the first statement of Proposition 9.2. {i2}

2. For $B_j = (G_{oo}^{(i)} - Q)$ we can first rewrite $\tilde{B}_j$ as

$$\tilde{B}_j = (\tilde{G}_{oo}^{(i)} - \tilde{Q}) = (\tilde{G}_{oo}^{(i)} - Y) + (Y - \tilde{Q}),$$

then expand according to Proposition 9.2. This gives that $\tilde{B}_j = \hat{B}_j + \mathcal{E}_j$, where

$$\hat{B}_j = \frac{\mathfrak{c}}{(d-1)^{\ell+1}} \sum_{\alpha \in A_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) + \frac{\mathfrak{c}}{(d-1)^{\ell+1}} \sum_{\alpha \neq \beta \in A_i} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{U}_j. \quad (11.14) \quad \{\mathfrak{e}:\text{zhankai1}\}$$

Here $\mathcal{U}_j$ is an $O(1)$ -weighted sum of terms of the form $(d-1)^{3(h-1)\ell} R_h$ with $h \geq 2$; $\mathcal{E}_j$ is $\mathcal{E} + ((Y - Q) + (Q - \tilde{Q}))$ in the first statement of Proposition 9.2.

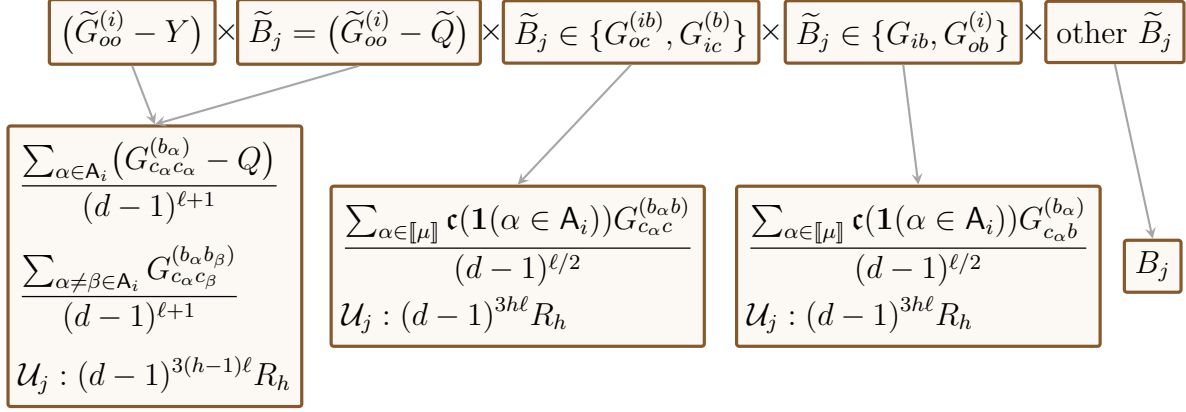


Figure 7: Schematic decomposition of the expression into three cases and products.

3. For $B_j \in \{G_{oc}^{(ib)}, G_{ic}^{(b)}\}_{(i,o) \neq (b,c) \in \mathcal{C}^\circ}$, by Proposition 9.3, we have $\tilde{B}_j = \hat{B}_j + \mathcal{E}_j$, where

$$\hat{B}_j = (d-1)^{-\ell/2} \sum_{\alpha \in [\mu]} \mathbf{c}_1(\mathbf{1}(\alpha \in A_i)) G_{c_\alpha c}^{(b_\alpha b)} + \mathcal{U}_j. \quad (11.15)$$

Here $\mathcal{U}_j$ is an $O(1)$ -weighted sum of terms of the form $(d-1)^{3h\ell} R_h G_{c_\alpha c}^{(b_\alpha b)}$ with $h \geq 1$; and $\mathcal{E}_j$ is $\mathcal{E}$ in the first statement of Proposition 9.3.

For $B_j \in \{G_{ib}, G_{ob}^{(i)}\}_{(i,o) \neq (b,c) \in \mathcal{C}^\circ}$, we have a similar expansion $\tilde{B}_j = \hat{B}_j + \mathcal{E}_j$, where

$$\hat{B}_j = (d-1)^{-\ell/2} \sum_{\alpha \in [\mu]} \mathbf{c}_1(\mathbf{1}(\alpha \in A_i)) G_{c_\alpha b}^{(b_\alpha)} + \mathcal{U}_j. \quad (11.16)$$

Here $\mathcal{U}_j$ is an $O(1)$ -weighted sum of terms of the form $(d-1)^{3h\ell} R_h G_{c_\alpha b}^{(b_\alpha)}$ with $h \geq 2$; $\mathcal{E}_j$ is $\mathcal{E}$ in the second statement of Proposition 9.3.

4. For $B_j \in \{G_{cc}^{(b)}, G_{cc'}^{(bb')}, G_{bc'}^{(b')}, G_{bb'}\}_{(b,c) \neq (b',c') \in \mathcal{C}^\circ}$, by Proposition 9.3, we have $\tilde{B}_j = \hat{B}_j + \mathcal{E}_j$, where $\hat{B}_j = \mathcal{U}_j = B_j$, and $\mathcal{E}_j$ is $\mathcal{E}$ in the third statements of Proposition 9.3.
5. For $B_j = Q - m_{\text{sc}}(z)$, we have by the statement (10.11) in Proposition 10.3, $\tilde{B}_j = \hat{B}_j + \mathcal{E}_j$, where $\hat{B}_j = \mathcal{U}_j = B_j$, and $|\mathcal{E}_j| \lesssim (d-1)^{6\ell} N^\circ \Phi$.

We can replace $(\tilde{G}_{oo}^{(i)} - Y) \prod_{j=1}^r \tilde{B}_j$ in (11.11) with $\prod_{j=0}^r \hat{B}_j$, with the overall error from this substitution being negligible.

Lemma 11.3. *Adopt the notation and assumptions in Proposition 11.2, we can rewrite (11.11) as*

$$(11.11) = \sum_{\mathbf{z} \in \mathcal{Z}_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \prod_{j=0}^r \hat{B}_j \right] + O(N^{-\mathfrak{b}/2} \mathbb{E}[\Psi]). \quad (11.17)$$

If we temporarily ignore the indicator and the averaging over embeddings, the total substitution error satisfies

$$(d-1)^{3r\ell} N^{-r\mathfrak{b}} \sum_{j=0}^r \mathbb{E}[|\mathcal{E}_j|]$$

which follows from the bound $|B_j| \leq N^{-\mathfrak{b}}$ in (8.18). Using the same argument as in (7.58), we obtain $\mathbb{E}[|\mathcal{E}_j|] \lesssim (d-1)^{2\ell} \mathbb{E}[\Psi]$ in all five cases above, except for Item 2, where we also have the additional errors

(ignoring the indicator functions)

$$\mathbb{E}[|Y - Q| + |\tilde{Q} - Q|] \lesssim N^{b/8} \mathbb{E}[\Psi] + \mathbb{E}[(d-1)^{6\ell} N^o \Phi], \quad (11.18)$$

where we used the definitions (4.22) and estimate (10.11). Finally, (11.17) follows from the fact that $r \geq 1$.

We can further decompose $(d-1)^{3r\ell} \prod_{j=0}^r \hat{B}_j$ in (11.17). The discussions in Item 1–Item 5 provide the decomposition of $(d-1)^{3r\ell} \prod_{j=0}^r \hat{B}_j$ as an $O(1)$ -weighted sum of terms of the following form:

$$\frac{(d-1)^{3\hat{h}\ell}}{(d-1)^{(k_1+(k_2+k_3)/2)\ell}} \sum_{\alpha, \beta, \gamma} \prod_{m=1}^{k_1} A_{\alpha_m} \prod_{m=1}^{k_2} B_{\beta_m} \prod_{m=1}^{k_3/2} C_{\gamma_{2m-1}\gamma_{2m}} R_{\hat{h}+1-k_1-k_2-k_3/2}. \quad (11.19) \quad \{\{e:oneterm0\}\}$$

Here the summation for α, β, γ runs through each $\alpha_m, \beta_m, \gamma_m$ in one of the sets $[\mu]$, A_i or $[\mu] \setminus A_i$ (we recall A_i from Figure 5). The factors in (11.19) are defined as follows:

$$A_\alpha = (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q), \quad B_\beta \in \left\{ G_{c_\beta c}^{(b_\beta b)}, G_{c_\beta b}^{(b_\beta)} \right\}_{(b,c) \in \mathcal{C}^o}, \quad C_{\gamma\gamma'} = G_{c_\gamma c_{\gamma'}}^{(b_\gamma b_{\gamma'})}. \quad (11.20) \quad \{\{e:defABC\}\}$$

Here A_α originates from $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$ in (11.13), (11.14) and B_β comes from $G_{c_\alpha c}^{(b_\alpha b)}, G_{c_\alpha b}^{(b_\alpha)}$ in (11.15), (11.16). $C_{\gamma\gamma'}$ are from $G_{c_\alpha c_\alpha}^{(b_\alpha b_\beta)}$ in (11.13), (11.14). In total there are $k_1 + k_2 + k_3/2$ such terms. $R_{\hat{h}+1-k_1-k_2-k_3/2}$ collects all R_h factors from $\mathcal{U}_j$ as in Item 1–Item 5, with a total count of $\hat{h} + 1 - k_1 - k_2 - k_3/2$. Thus, the summand in (11.19) consists of $\hat{h} + 1$ factors in total. We notice that after expansion, each $\hat{B}_j$ gives at least one term, so $\hat{h} + 1 \geq r + 1$. Finally the coefficient $(d-1)^{3\hat{h}\ell}$ arises from the following crucial observation: in the replacements outlined in Item 1–Item 5, each term $(d-1)^{3\ell} B_j$ is replaced by one of the terms $(d-1)^{3\ell} A_\alpha, (d-1)^{3\ell} A_\beta, (d-1)^{3\ell} C_{\gamma\gamma'}$ as in (11.20), or a factor R_h (as in (11.12)) for $h \geq 1$ with coefficient at most $(d-1)^{3h\ell}$. This leads to the following statement

Claim 11.4. (11.17) can be written as an $O(1)$ -weighted sum of terms of the following form: For $\hat{h} \geq r$,

$$\frac{1}{(d-1)^{(k_1+(k_2+k_3)/2)\ell}} \sum_{\alpha, \beta, \gamma} \sum_{i^+} \frac{(d-1)^{3\hat{h}\ell}}{Z_{\mathcal{F}^+}} \times \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \hat{R}_{i^+} \right], \quad (11.21) \quad \{\{e:oneterm\}\}$$

$$\hat{R}_{i^+} = R_{\hat{h}+1} = \prod_{m=1}^{k_1} A_{\alpha_m} \prod_{m=1}^{k_2} B_{\beta_m} \prod_{m=1}^{k_3/2} C_{\gamma_{2m-1}\gamma_{2m}} E(\chi).$$

Here $R_{\hat{h}+1}$ is the summand in (11.19). ~~It is the term~~ $\hat{R}_{i^+} \in \text{Adm}(\hat{h}+1, \mathcal{F}^+, \mathcal{G})$. The summation for α, β, γ is over each $\alpha_m, \beta_m, \gamma_m$ in one of the sets $[\mu]$, A_i or $[\mu] \setminus A_i$; and the factors in (11.21) are given by (11.20), and $E(\chi) = R_{\hat{h}+1-k_1-k_2-k_3/2}$ is a product of the remaining terms which depends on $\{b_\chi, c_\chi\}_{\chi \in \mathcal{X}}$ for some $\mathcal{X} \subset [\mu]$.

The summation over α, β, γ in (11.21) produces $(d-1)^{(k_1+k_2+k_3)\ell}$ terms of the form (11.4). Note that this number is much larger than the normalization factor $(d-1)^{(k_1+(k_2+k_3)/2)\ell}$ in the denominator. However, as we will show in Lemma 11.6, most of these terms can be seen almost immediately to be negligible. The cancellation mechanism is similar to those in (7.31) and (7.38). Before stating Lemma 11.6, we first introduce some notation.

We view $\alpha, \beta, \gamma, \chi$ as words, which are sequences of indices in $[\mu]$. In particular (α, β, γ) is a word with length $k_1 + k_2 + k_3$. Given χ , we partition words $\omega \in [\mu]^{k_1+k_2+k_3}$ into equivalence classes. Two words $\omega \sim \omega'$ are equivalent if there is a bijection on $[\mu]$ which preserves χ and maps ω to ω' . We remark that the expectation in (11.21) depends only on the equivalence class of (α, β, γ) .

For any $f_0, f_1 \geq 0$, let $W(f_0, f_1)$ denote a set of representatives for equivalence classes of $[\mu]^{k_1+k_2+k_3}$. Here, for a word $\omega \in W(f_0, f_1)$, f_0 is the number of distinct indices (ignoring multiplicity) that do not appear in χ , and f_1 is the number of these indices appearing exactly once in ω . The length of ω is $k_1 + k_2 + k_3$, and $f_0 - f_1$ of these distinct indices appear at least twice in ω . This implies

$$k_1 + k_2 + k_3 - f_1 \geq 2(f_0 - f_1) \Rightarrow k_1 + k_2 + k_3 \geq 2f_0 - f_1. \quad (11.22) \quad \{\{e:f0f1\}\}$$

Example 11.5. The following are two possible terms of $\widehat{R}_{i+}$:

$$\begin{aligned} A_1 A_2 B_2 C_{47} C_{37} G_{c_{8c}}^{b_{8b}}(G_{c_4 c_4}^{(b_4)} - Q), \quad \omega = (\alpha, \beta, \gamma) = (1, 2), (2), (4, 7, 3, 7), \quad \chi = (8, 4), \\ A_1 A_5 B_5 C_{49} C_{39} G_{c_{8c}}^{b_{8b}}(G_{c_4 c_4}^{(b_4)} - Q), \quad \omega' = (\alpha', \beta', \gamma') = (1, 5), (5), (4, 9, 3, 9), \quad \chi = (8, 4). \end{aligned}$$

In this example, $\omega \sim \omega'$ (by mapping 2 to 5 and 7 to 9, and keeping all other indices in $\llbracket \mu \rrbracket$ fixed). By permutation invariance of the vertices, the two terms above have the same expectation. Moreover, in ω the indices 1, 2, 3, 7 do not appear in $\chi = (8, 4)$, and among them 1 and 3 appear only once, so $\omega \in W(4, 2)$.

The expectation in (11.21) depends only on the equivalence class of (α, β, γ) . Moreover, for fixed f_0 and f_1 , the summation of $(\alpha, \beta, \gamma) \sim \omega$, contains $O((d-1)^{f_0 \ell})$ terms. Thus for the summation over (α, β, γ) in (11.21), we can first sum over the equivalence classes. We recall the summand from (11.21)

$$\sum_{i+} \frac{(d-1)^{3\hat{h}\ell}}{Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \widehat{R}_{i+}(\alpha, \beta, \gamma, \chi) \right]. \quad (11.23) \quad \{\mathbf{e}:\text{partition}\}$$

We then have

$$\begin{aligned} (11.21) &= \frac{1}{(d-1)^{(k_1+(k_2+k_3)/2)\ell}} \sum_{\alpha, \beta, \gamma} (11.23) \quad \{\mathbf{e}:\text{partition}\} \\ &= \frac{1}{(d-1)^{(k_1+(k_2+k_3)/2)\ell}} \sum_{f_0, f_1} \sum_{\omega \in W(f_0, f_1)} \sum_{(\alpha, \beta, \gamma) \sim \omega} (11.23), \quad \{\mathbf{e}:\text{partition}\} \quad (11.24) \quad \{\mathbf{e}:\text{partition}\} \\ &= \sum_{f_0, f_1} \sum_{(\alpha, \beta, \gamma) \in W(f_0, f_1)} \frac{O((d-1)^{f_0 \ell})}{(d-1)^{(k_1+(k_2+k_3)/2)\ell}} \times (11.23). \quad \{\mathbf{e}:\text{partition}\} \end{aligned}$$

The following lemma states that for given α, β, γ , the summands in the last term of (11.24) is either negligible, or it can be reduced to a term as in (11.26) below, where each index in α, β, γ appears at least twice in R'_{i+} . \{1:erbu\}

Lemma 11.6. Fix $0 \leq f_1 \leq f_0$ satisfying (11.22), and a word $(\alpha, \beta, \gamma) \in W(f_0, f_1)$. Let $\mathbf{l}_{\text{single}} \subset \llbracket \mu \rrbracket$ denote the set of indices that appear only once among (α, β, γ) , and do not appear in χ . Then $|\mathbf{l}_{\text{single}}| = f_1$, and

$$\frac{(d-1)^{f_0 \ell}}{(d-1)^{(k_1+(k_2+k_3)/2)\ell}} \sum_{i+} \frac{(d-1)^{3\hat{h}\ell}}{Z_{\mathcal{F}^+}} \times \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) \widehat{R}_{i+}(\alpha, \beta, \gamma, \chi) \right] \quad (11.25) \quad \{\mathbf{e}:\text{oneterm1}\}$$

satisfies

1. If there exists $\alpha_m \in \mathbf{l}_{\text{single}}$ then $(11.25) = O(N^{-b/4} \mathbb{E}[\Psi])$.
2. Otherwise,

$$(11.25) = \frac{(d-1)^{3(\hat{h}+f_1)\ell}}{(d-1)^{\mathfrak{q}^++\ell/2} Z_{\mathcal{F}^+}} \sum_{i+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (d-1)^{-f_1/2} R'_{i+} \right] + O(N^{-b/4} \mathbb{E}[\Psi]), \quad (11.26) \quad \{\mathbf{e}:\text{oneterm2}\}$$

where $\mathfrak{q}^+ = k_1/2 + (k_1 + k_2 + k_3 - 2f_0 + 6f_1) \geq 0$, and $R'_{i+} \in \text{Adm}(\widehat{h} + f_1 + 1, \mathcal{F}^+, \mathcal{G})$ is obtained from $\widehat{R}_{i+}$ by making the following substitutions:

$$\begin{aligned} G_{c_{\beta_m} c}^{(b_{\beta_m} b)} &\Rightarrow G_{b_{\beta_m} c}^{(b)} (G_{c_{\beta_m} c_{\beta_m}}^{(b_{\beta_m})} - Q), \quad G_{c_{\beta_m} b}^{(b_{\beta_m})} \Rightarrow G_{b_{\beta_m} b} (G_{c_{\beta_m} c_{\beta_m}}^{(b_{\beta_m})} - Q), \quad \beta_m \in \mathbf{l}_{\text{single}}, \\ G_{c_{\gamma_{2m-1}} c_{\gamma_{2m}}}^{(b_{\gamma_{2m-1}} b_{\gamma_{2m}})} &\Rightarrow G_{b_{\gamma_{2m-1}} c_{\gamma_{2m}}}^{(b_{\gamma_{2m}})} (G_{c_{\gamma_{2m-1}} c_{\gamma_{2m-1}}}^{(b_{\gamma_{2m-1}})} - Q), \quad \gamma_{2m-1} \in \mathbf{l}_{\text{single}}, \quad \gamma_{2m} \notin \mathbf{l}_{\text{single}}, \\ G_{c_{\gamma_{2m-1}} c_{\gamma_{2m}}}^{(b_{\gamma_{2m-1}} b_{\gamma_{2m}})} &\Rightarrow G_{c_{\gamma_{2m-1}} b_{\gamma_{2m}}}^{(b_{\gamma_{2m-1}})} (G_{c_{\gamma_{2m}} c_{\gamma_{2m}}}^{(b_{\gamma_{2m}})} - Q), \quad \gamma_{2m-1} \notin \mathbf{l}_{\text{single}}, \quad \gamma_{2m} \in \mathbf{l}_{\text{single}}, \\ G_{c_{\gamma_{2m-1}} c_{\gamma_{2m}}}^{(b_{\gamma_{2m-1}} b_{\gamma_{2m}})} &\Rightarrow G_{b_{\gamma_{2m-1}} b_{\gamma_{2m}}} (G_{c_{\gamma_{2m-1}} c_{\gamma_{2m-1}}}^{(b_{\gamma_{2m-1}})} - Q) (G_{c_{\gamma_{2m}} c_{\gamma_{2m}}}^{(b_{\gamma_{2m}})} - Q), \quad \gamma_{2m-1}, \gamma_{2m} \in \mathbf{l}_{\text{single}}. \end{aligned} \quad (11.27) \quad \{\mathbf{e}:\text{final_rep}\}$$

The first statement in Lemma 11.6 extends (7.30), and the first statement in (11.27) extends (7.32). Since the underlying ideas are analogous, we omit the proof. The first three claims in (11.27) essentially follow from the following estimates:

$$\begin{aligned}
& \frac{1}{Nd} \sum_{b_\alpha, c_\alpha \in \llbracket N \rrbracket} A_{c_\alpha b_\alpha} A_{bc} G_{c_\alpha c}^{(b_\alpha b)} \\
&= \frac{1}{Nd} \sum_{b_\alpha, c_\alpha \in \llbracket N \rrbracket} A_{c_\alpha b_\alpha} A_{bc} \frac{G_{b_\alpha c}^{(b)}}{\sqrt{d-1}} (G_{c_\alpha c_\alpha}^{(b)} - Q) + \text{“negligible error,”} \\
& \frac{1}{Nd} \sum_{b_\alpha, c_\alpha \in \llbracket N \rrbracket} A_{c_\alpha b_\alpha} A_{bc} G_{c_\alpha b}^{(b_\alpha)} \\
&= \frac{1}{Nd} \sum_{b_\alpha, c_\alpha \in \llbracket N \rrbracket} A_{c_\alpha b_\alpha} A_{bc} \frac{G_{b_\alpha b}^{(b_\alpha)}}{\sqrt{d-1}} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) + \text{“negligible error,”}
\end{aligned} \tag{11.28}$$

which can be proven in the same manner as (7.38). Again, we omit the proof. We can repeat the above substitutions for all f_1 indices in $\mathbf{l}_{\text{single}}$, which appear only once among (α, β, γ) and do not appear in χ . Each substitution contributes an extra factor, and after these substitutions

$$\widehat{R}_{\mathbf{i}^+} \Rightarrow (d-1)^{-f_1/2} R'_{\mathbf{i}^+}, \quad R'_{\mathbf{i}^+} \in \text{Adm}(\widehat{h} + f_1 + 1, \mathcal{F}^+, \mathcal{G}), \tag{11.29}$$

and

$$\mathbf{q}^+ = 2k_1 + k_2 + k_3 - 2f_0 + 6f_1 = k_1/2 + (k_1 + k_2 + k_3 - 2f_0 + 6f_1).$$

We remark that in (11.26), if $\mathbf{q}^+ \geq 1$, we gain an additional factor of $(d-1)^{-\ell/2}$. We do not obtain this extra factor only if $k_1 = 0$, $f_1 = 0$ and each index in (α, β, γ) appears exactly twice without appearing in χ (so $k_2 + k_3 = 2f_0$).

Proof of (11.3) in Proposition 11.2. Up to a negligible error, the expression (11.3) can be rewritten as an $O(1)$ -weighted sum of terms in the form of (11.25). We also refer back to the more explicit expression given in (11.19) and (11.21). If the assumptions in the first statement in Lemma 11.6 hold, there is nothing to prove. So in the rest of the proof we can focus on the second case (11.26).

There are several cases in which we can apply (11.26), based on the decomposition of $\widehat{B}_0$ (as in (11.13)) using terms $m_{\text{sc}}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \in \mathbf{A}_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$, $m_{\text{sc}}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \neq \beta \in \mathbf{A}_i} (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} - Q)$ or a term in $\mathcal{U}_0$. We treat each of these separately.

1. Assume $\widehat{R}_{\mathbf{i}^+}$ in (11.21) contains the factor $m_{\text{sc}}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \in \mathbf{A}_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$ from the decomposition of $\widehat{B}_0$, then in (11.26), $k_1 \geq 1$ and $R'_{\mathbf{i}^+}$ contains a factor $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$. Let $R'_{\mathbf{i}^+} = (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) R_{\mathbf{i}^+}$, then $R_{\mathbf{i}^+} \in \text{Adm}(\widehat{h} + f_1, \mathcal{F}^+, \mathcal{G})$ (recall from (11.26)). We claim that replacing Q with Y yields a negligible error. To see this, we write

$$\begin{aligned}
& \frac{(d-1)^{3(\widehat{h}+f_1)\ell}}{(d-1)^{\mathbf{q}^+\ell/2} Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) R_{\mathbf{i}^+} \right] \\
&= \frac{(d-1)^{3(\widehat{h}+f_1)\ell}}{(d-1)^{\mathbf{q}^+\ell/2} Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) R_{\mathbf{i}^+} \right] \\
&+ \frac{(d-1)^{3(\widehat{h}+f_1)\ell}}{(d-1)^{\mathbf{q}^+\ell/2} Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (Q - Y) R_{\mathbf{i}^+} \right].
\end{aligned} \tag{11.30}$$

Note that $k_1 \geq 1$, so $q^+ \geq 1$, and the second term on the right-hand side of (11.30) is bounded by

$$\begin{aligned} & \frac{(d-1)^{3(\hat{h}+f_1)\ell}}{(d-1)^{q^+\ell/2} Z_{\mathcal{F}^+}} \sum_{i^+} \mathbb{E} [I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |Q - Y| |R_{i^+}|] \\ & \lesssim \frac{(d-1)^{3(\hat{h}+f_1)\ell} N^{-(\hat{h}+f_1)b}}{(d-1)^{q^+\ell/2}} \mathbb{E} [\mathbf{1}(\mathcal{G} \in \Omega) |Q - Y|] \lesssim N^{-b/2} \mathbb{E}[\Psi], \end{aligned}$$

where in the first inequality we used (8.18); in the second inequality we used (4.22) and $\hat{h} \geq r \geq 2$. The first term on the right-hand side of (11.30) is in the form of (11.4), by setting $r^+ = \hat{h} + f_1 \geq \hat{h} \geq r$. {ii:offab}

2. If $\hat{R}_{i^+}$ in (11.21) contains the factor $m_{\text{sc}}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \neq \beta \in A_i} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ from the decomposition of $\hat{B}_0$, then $k_3 \geq 2$ in (11.26). There are several cases for the factor $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ contained in $\hat{R}_{i^+}$.

If in $\hat{R}_{i^+}$ (from (11.21)) $\alpha \in \mathbf{l}_{\text{single}}$, then $f_1 \geq 1$ in (11.26). Also, (11.27) implies that R'_{i^+} contains $G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q$. By the same argument as in (11.30), this leads to (11.4) by setting $r^+ = \hat{h} + f_1 \geq r + 1$. The same conclusion holds if $\beta \in \mathbf{l}_{\text{single}}$.

In the remaining cases $\alpha, \beta \notin \mathbf{l}_{\text{single}}$. There are again two cases: either $\hat{R}_{i^+}$ in (11.21) contains at least two terms in the form $\{G_{cc'}^{(bb')}, G_{cb'}^{(b)}, G_{bb'}^{(b)}\}_{(b,c) \neq (b',c') \in \mathcal{K}^+}$, or $\hat{R}_{i^+}$ in (11.21) contains both factors $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q), (G_{c_\beta c_\beta}^{(b_\beta)} - Q)$.

In the first case, if $\hat{R}_{i^+}$ in (11.21) contains at least two terms in the form $\{G_{cc'}^{(bb')}, G_{cb'}^{(b)}, G_{bb'}^{(b)}\}_{(b,c) \neq (b',c') \in \mathcal{K}^+}$, so does (11.26). By our assumption $r \geq 2$, we have $\hat{h} + f_1 + 1 \geq \hat{h} + 1 \geq r + 1 \geq 3$. We remark that this is the only point in the argument where the assumption $r \geq 2$ is required; all other parts of the proof remain valid for $r \geq 1$. Then (8.19) (with r taking value $\hat{h} + f_1 + 1 \geq 3$) implies that (11.26) is bounded by $O(N^{-b/2} \mathbb{E}[\Psi])$.

In the second case, $\hat{R}_{i^+}$ in (11.21) contains factors $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q), (G_{c_\beta c_\beta}^{(b_\beta)} - Q)$. Then they are either contained in $E(\chi)$ or $k_1 \geq 1$. In both cases $q^+ \geq 1$ in (11.26). The same argument as in (11.30) leads to (11.4), by setting $r^+ = \hat{h} + f_1 \geq \hat{h} \geq r$.

3. In the remaining case, $\hat{R}_{i^+}$ in (11.21) contains a factor $(d-1)^{3(h-1)\ell} R_h$ from $\mathcal{U}_0$ in the decomposition (11.13) of $\hat{B}_0$. Here $h \geq 2$, R_h is an S -product term (recall from Definition 9.1), and it contains at least one factor of the form $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$ or $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$. Moreover, in this case the factor R_h is included in $E(\chi)$ in (11.21), and $\hat{h} \geq r + (h-1) \geq r + 1$.

If R_h contains at least one term of the form $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$, by the same argument as in (11.30), (11.26) leads to (11.4) with $r^+ = \hat{h} + f_1 \geq \hat{h} \geq r + 1$.

In the other cases, R_h contains at least one term of the form $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$. By the same argument as in the second statement of Lemma 11.6, if $\{b_\alpha, c_\alpha\}$ do not appear in other terms of $\hat{R}_{i^+}$ (but $\{b_\beta, c_\beta\}$ do appear), we can replace $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ by $G_{b_\alpha c_\beta}^{(b_\beta)} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) / \sqrt{d-1}$; if $\{b_\beta, c_\beta\}$ do not appear in other terms of $\hat{R}_{i^+}$ (but $\{b_\alpha, c_\alpha\}$ do appear), we can replace $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ by $G_{b_\beta c_\alpha}^{(b_\alpha)} (G_{c_\beta c_\beta}^{(b_\beta)} - Q) / \sqrt{d-1}$; and if $\{b_\alpha, c_\alpha, b_\beta, c_\beta\}$ do not appear in other terms of $\hat{R}_{i^+}$, we can replace it by $G_{b_\alpha b_\beta}^{(b_\alpha)} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) (G_{c_\beta c_\beta}^{(b_\beta)} - Q) / (d-1)$. Moreover, the errors from such replacements are bounded by $O(N^{-b/4} \mathbb{E}[\Psi])$. Then by the same argument as in Item 2, either (11.26) is bounded by $O(N^{-b/4} \mathbb{E}[\Psi])$, or (11.26) leads to (11.4) with $r^+ \geq \hat{h} + f_1 \geq \hat{h} \geq r + 1$. □

Proof of (11.5) in Proposition 11.2. Up to a negligible error, the expression (11.5) can also be rewritten as an $O(1)$ -weighted sum of terms in the form of (11.25). We also refer back to the more explicit expression

given in (11.19) and (11.21). If the assumptions in the first statement in Lemma 11.6 hold, there is nothing to prove. So in the rest of the proof we can focus on the second case (11.26).

We recall that in (11.5), $r = 1$ and $B_1 = (G_{oo}^{(i)} - Q)$. If $\hat{h} + f_1 \geq 2$ (from (11.26)), we can proceed in exactly the same manner as in the proof of (11.3). Otherwise, $\hat{h} = 1$ and $f_1 = 0$. We assume this scenario in the following discussion.

1. Assume $\hat{R}_{i+}$ in (11.21) contains $m_{sc}^{2\ell+2}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \in A_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$ from the decomposition (11.13) of $\hat{B}_0$. Since $\hat{h} = 1, f_1 = 0$, $\hat{R}_{i+}$ also contains $m_{sc}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \in A_i} (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$ from the decomposition of $\hat{B}_1$ (we recall the precise coefficients from (9.1)). Then (11.26) is of the form

$$\frac{m_{sc}^{4(\ell+1)}(z)}{(d-1)^{2(\ell+1)}} \sum_{\alpha \in A_i} \sum_{i^+} \frac{1}{Z_{\mathcal{F}^+}} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)^2]. \quad (11.31) \quad \{\mathbf{e:main1}\}$$

After replacing a copy of $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q)$ by $(G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y)$, (11.31) is a $\mathbf{q}^+$ -weighted sum of terms in the form (11.6) with $\mathbf{q}^+ = 2$, and the error is bounded by $O(N^{-b/2} \mathbb{E}[\Psi])$.

2. If $\hat{R}_{i+}$ in (11.21) contains $m_{sc}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \neq \beta \in A_i} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ from the decomposition (11.13) of $\hat{B}_0$. Since $\hat{h} = 1, f_1 = 0$, $\hat{R}_{i+}$ also contains $m_{sc}^{2(\ell+1)}(z)(d-1)^{-(\ell+1)} \sum_{\alpha \neq \beta \in A_i} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ from the decomposition of $\hat{B}_1$ (we recall the precise coefficients from (9.1)). Then (11.26) is of the form

$$\frac{2m_{sc}^{4(\ell+1)}(z)}{(d-1)^{2(\ell+1)}} \sum_{\alpha \neq \beta \in A_i} \sum_{i^+} \frac{1}{Z_{\mathcal{F}^+}} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2]. \quad (11.32) \quad \{\mathbf{e:main2}\}$$

By (8.19) with $\mathbf{q}^+ = 2$, the quantity in (11.32) is bounded by $O(N^0 \mathbb{E}[\Psi])$, where we also used that the factor $(d-1)^{-2(\ell+1)}$ is canceled by the summation over $\alpha \neq \beta$, which contains $O((d-1)^{2(\ell+1)})$ terms.

3. In the remaining case, $\hat{R}_{i+}$ in (11.21) contains a factor $(d-1)^{3(h-1)\ell} R_h$ with $h \geq 2$ from $\mathcal{U}_0$ in the decomposition (11.13) of $\hat{B}_0$. Then $\hat{h} \geq r+1 \geq 2$.

□

12 Proof of the first loop equation

In this section, we prove the first loop equation (4.25). We need to identify the leading order error terms from Proposition 7.3 and Proposition 11.2. These refined estimates are presented in the following three propositions.

Proposition 12.1. *Adopt the notation and assumptions in Proposition 7.3, and define the index set $A_i := \{\alpha \in [\mu] : \text{dist}_{\mathcal{T}}(i, l_\alpha) = \ell + 1\}$ (see Figure 5). We recall the local Green's functions L and $L^{(i)}$ (with vertex i removed) from (8.13) and (8.16). I_1 in (7.8) explicitly given by*

$$I_1 = \sum_{\alpha \in A_i} \sum_{i^+} \frac{m_{sc}^{2(\ell+1)}(z) L_{l_\alpha l_\alpha}^{(i)}}{(d-1)^{\ell+2} Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Y) (G_{c_\alpha c_\alpha}^{(b_\alpha)} - Q) \right]. \quad (12.1) \quad \{\mathbf{e:case1_erm}\}$$

We have the following refined expression for the error term $\mathcal{E}$ in (7.8) $\{\mathbf{e:case1_erm}\}$

$$\mathcal{E} = (12.3) + \{(12.4) \text{ error term}\} \quad (12.2) \quad \{\mathbf{e:daerrorE}\}$$

where

$$(12.3) = \sum_{\alpha, \beta \in \mathbf{A}_i} \sum_{\mathbf{i}^+} \frac{m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)^{\ell+1} Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) (\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}) \right], \quad (12.3) \quad \{\mathbf{e}:\text{first_erm}\}$$

$$(12.4) = \sum_{\substack{\alpha \in \mathbf{A}_i, \beta \in \llbracket \mu \rrbracket \\ \alpha \neq \beta}} \sum_{\mathbf{i}^+} \frac{m_{\text{sc}}^{2(\ell+1)}(z) (L_{l_\beta l_\beta}^{(i)} + L_{l_\alpha l_\beta}^{(i)})}{(d-1)^{\ell+2} Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2 \right], \quad (12.4) \quad \{\mathbf{e}:\text{second_erm}\}$$

Proposition 12.2. *Adopt the notation and assumptions in Proposition 11.2. The error from expanding I_1 (from (12.1)) ~~is given by~~ $\mathbf{p}:\text{track_error}$*

$$\left(1 - \left(\frac{m_{\text{sc}}(z)}{\sqrt{d-1}} \right)^{2\ell+2} \right) \frac{2m_d(z)m_{\text{sc}}^{6(\ell+1)}(z)}{(d-1)^{2\ell+3}} \sum_{\alpha \neq \beta \in \mathbf{A}_i} \sum_{\mathbf{i}^+} \frac{1}{Z_{\mathcal{F}^+}} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2] + O(N^{-b/4} \mathbb{E}[\Psi]). \quad (12.5) \quad \{\mathbf{e}:\text{track_error}\}$$

For z close to the spectral edge ± 2 , the following proposition gives refined estimates for the error terms in (12.3), ~~(12.4)~~ $\{\mathbf{l}:\text{first_erm}\}$

Proposition 12.3. *Adopt the notation and assumptions in Proposition 7.3, and recall $\mathcal{A}$ from (1.3). For $z \in \mathbf{D}$ (recall from (1.8)) and $|z - 2| \leq N^{-\mathfrak{q}}$, we have the following estimates for the terms involved in the error (12.2):*

$$(12.3) = \left\{ \left(\frac{d(d-1)^\ell}{d-2} \right) \frac{d}{d-2} \right\} \frac{1}{\mathcal{A}^2} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \frac{\partial_z m_N(z)}{N} \right] + O \left(\frac{\mathbb{E}[\Psi]}{(d-1)^\ell} \right), \quad (12.6) \quad \{\mathbf{e}:\text{first_erm}\}$$

$$(12.4) = \left\{ \left(\frac{d+2}{d-2} \frac{d(d-1)^\ell}{d-2} \right) (\ell+1) \right\} \frac{1}{\mathcal{A}^2} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \frac{\partial_z m_N(z)}{N} \right] + O \left(\frac{\mathbb{E}[\Psi]}{(d-1)^\ell} \right), \quad (12.7) \quad \{\mathbf{e}:\text{refine_Gccerror}\}$$

Moreover, the error (12.5) ~~is given by~~ $\{\mathbf{e}:\text{refine_Gccerror}\}$

$$(12.5) = \left\{ \left(\frac{2}{d-2} \frac{1}{\mathcal{A}^2} \right) \right\} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \frac{\partial_z m(z)}{N} \right] + O \left(\frac{\mathbb{E}[\Psi]}{(d-1)^\ell} \right). \quad (12.8) \quad \{\mathbf{e}:\text{GIGG3}\}$$

If $|z + 2| \leq N^{-\mathfrak{q}}$, analogous statements hold after multiplying the right-hand sides by -1 .

Proof of Theorem 4.5. We will prove (4.25) ~~only for $|z - 2| \leq N^{-\mathfrak{q}}$, the other case $|z + 2| \leq N^{-\mathfrak{q}}$, can be established in exactly the same way.~~ $\{\mathbf{e}:\text{GIGG3}\}$ To prove (4.25), we must track the errors from the iteration process more carefully. These refined error estimates are presented in Proposition 12.1, Proposition 12.2 and Proposition 12.2. By adding (12.6) and (12.7), the error $\mathcal{E}$ from Proposition 7.3 is given by

$$- \left(\ell + 1 - \frac{2}{d-2} \right) \frac{1}{\mathcal{A}^2} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \frac{\partial_z m_N(z)}{N} \right] + O \left(\frac{\mathbb{E}[\Psi]}{(d-1)^\ell} \right). \quad (12.9) \quad \{\mathbf{e}:\text{final_error}\}$$

From Proposition 11.2, the error from expanding (11.3) is ~~is bounded by $O(N^{-b/4} \mathbb{E}[\Psi])$.~~ $\{\mathbf{e}:\text{final_error}\}$ The errors from expanding (11.5) with $\mathfrak{q} \geq 1$ are bounded by $O((d-1)^{-\ell/2} N^{\mathfrak{q}} \mathbb{E}[\Psi])$. For $\mathfrak{q} = 0$, the error from expanding (11.5) is given in (12.5) and (12.8)

$$- \frac{2}{d-2} \frac{1}{\mathcal{A}^2} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \frac{\partial_z m_N(z)}{N} \right] + O \left(\frac{\mathbb{E}[\Psi]}{(d-1)^\ell} \right). \quad (12.10) \quad \{\mathbf{e}:\text{final_error}\}$$

The correction terms in the microscopic loop equation (4.25) is ~~obtained by summing the refined errors from (12.9) and (12.10).~~ $\{\mathbf{e}:\text{final_error}\}$

□

Proof of Proposition 12.1. The first statement (12.1) is from (7.49). The decomposition of the error $\mathcal{E}$ as in (12.2) is from the first term in (7.58) and J_2 in (7.46). $\square$

Proof of Proposition 12.2. We recall I_1 from (12.1). Conditioned on $I(\mathcal{F}^+, \mathcal{G}) = 1$, the expectation in (12.1) does not depend on α . Moreover, $|\mathbf{A}_i| = (d-1)^{\ell+1}$, and by (3.2), $L_{l_\alpha l_\alpha}^{(i)} = m_d(z)(1 - (-m_{\text{sc}}(z)/\sqrt{d-1})^{2\ell+2})$. We denote $(i, o) = (b_\alpha, c_\alpha)$ and $(\mathcal{F}, \mathbf{i}) = (\mathcal{F}^+, \mathbf{i}^+)$, and rewrite I_1 from (12.1) as

$$\left(1 - \left(\frac{m_{\text{sc}}(z)}{\sqrt{d-1}}\right)^{2\ell+2}\right) \frac{m_d(z)m_{\text{sc}}^{2(\ell+1)}(z)}{(d-1)Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[I(\mathcal{F}, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{oo}^{(i)} - Y)(G_{oo}^{(i)} - Q) \right], \quad (12.11) \quad \{\text{e:I1_rewrite}\}$$

which is in the form of (11.5), up to the coefficient.

From the proof of (11.5) in Proposition 11.2, the errors from expanding (11.5) are either bounded by $O(N^{-b/4}\mathbb{E}[\Psi])$, or given by (11.32). Thus, the error from expanding (12.11) is given by

$$\left(1 - \left(\frac{m_{\text{sc}}(z)}{\sqrt{d-1}}\right)^{2\ell+2}\right) \frac{2m_d(z)m_{\text{sc}}^{6(\ell+1)}(z)}{(d-1)^{2\ell+3}} \sum_{\alpha \neq \beta \in \mathbf{A}_i} \sum_{\mathbf{i}^+} \frac{1}{Z_{\mathcal{F}^+}} \mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2] + O(N^{-b/4}\mathbb{E}[\Psi]).$$

This finishes the proof of (12.5). $\square$

Proof of Proposition 12.3. To illustrate the basic ideas, we will only prove (12.7).

First we notice that the expectation does not depend on (α, β) . Thus we can first sum over $\alpha \in \mathbf{A}_i, \alpha \neq \beta$ for the coefficient

$$\sum_{\substack{\alpha \in \mathbf{A}_i, \beta \in [\mu] \\ \alpha \neq \beta}} \frac{m_{\text{sc}}^{2(\ell+1)}(z)(L_{l_\beta l_\beta}^{(i)} + L_{l_\alpha l_\alpha}^{(i)})}{(d-1)^{\ell+2}} = \left(\frac{d+2}{d-2} - \frac{d(d-1)^\ell}{d-2} - (\ell+1) + O((d-1)^{-\ell}) \right) \quad (12.12) \quad \{\text{e:thesum}\}$$

We notice that for $|z \pm 2| \leq N^{-g}$,

$$\begin{aligned} m_{\text{sc}}(z) &= -1 + O(\sqrt{|z-2|}) = -1 + O(N^{-g/2}), \\ m_d(z) &= -\frac{d-1}{d-2} + O(\sqrt{|z-2|}) = -\frac{d-1}{d-2} + O(N^{-g/2}). \end{aligned} \quad (12.13) \quad \{\text{e:mscmd}\}$$

Moreover, when $I(\mathcal{F}^+, \mathcal{G}) = 1$, vertex o has radius $\mathfrak{R}$ tree neighborhood. Then $\mu = d(d-1)^\ell$, $|\mathbf{A}_i| = (d-1)^{\ell+1}$, $L_{l_\beta l_\beta}^{(i)}$ and $L_{l_\alpha l_\alpha}^{(i)}$ are given explicitly by the Green's function of trees from Proposition 3.3. Equation (12.12) follows from direct computation, we omit it.

By plugging (12.12) into (12.11), we get

$$(12.4) = \left\{ \left(\frac{d+2}{d-2} - \frac{d(d-1)^\ell}{d-2} - (\ell+1) + O((d-1)^{-\ell}) \right) \sum_{\mathbf{i}^+} \frac{1}{Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2 \right] \right\} \quad (12.14) \quad \{\text{e:plugin}\}$$

Next we show

$$\sum_{\mathbf{i}^+} \frac{1}{Z_{\mathcal{F}^+}} \mathbb{E} \left[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) (G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)})^2 \right] = \frac{1}{\mathcal{A}^2} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) \frac{\partial_z m(z)}{N} \right] + O(N^{-b/2}\mathbb{E}[\Psi]), \quad (12.15) \quad \{\text{e:averageGG}\}$$

and (12.7) follows from combining (12.14) and (12.15)

If we temporarily ignore the indicator and the averaging over embeddings, writing $(b_\alpha, c_\alpha, b_\beta, c_\beta)$ as (b, c, b', c') , the above statement (12.15) reduces to computing

$$\frac{1}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} (G_{cc'}^{(bb')})^2 = \frac{1}{\mathcal{A}^2} \frac{\partial_z m(z)}{N} + O(N^{-b/2}\Phi). \quad (12.16) \quad \{\text{e:remove_in}\}$$

We start with the Schur complement formula (2.4) `{e:remove_indices}`

$$G_{cc'}^{(bb')} = G_{cc'} - (G(G|_{\{bb'\}})^{-1}G)_{cc'}, \quad (G|_{\{bb'\}})^{-1} = \frac{1}{G_{bb}G_{b'b'} - G_{bb'}^2} \begin{bmatrix} G_{b'b'} & -G_{bb'} \\ -G_{bb'} & G_{bb} \end{bmatrix} \quad (12.17)$$

Conditioned on $I(\mathcal{F}^+, \mathcal{G}) = 1$, by (3.21) and (12.13), the terms

$$\begin{aligned} G_{bc} - \frac{\sqrt{d-1}}{d-2}, \quad G_{b'c'} - \frac{\sqrt{d-1}}{d-2}, \quad G_{bb'}, \quad G_{bc'}, \quad G_{cb'}, \quad G_{cc'} \\ G_{bb} + \frac{d-1}{d-2}, \quad G_{cc} + \frac{d-1}{d-2}, \quad G_{b'b'} + \frac{d-1}{d-2}, \quad G_{c'c'} + \frac{d-1}{d-2}, \end{aligned} \quad (12.18) \quad \{\text{e:centered}\}$$

are all bounded by $O(N^{-b})$. The Schur complement formula (2.5) `{e:centered_erm}`

$$\begin{aligned} G_{cc'}^{(bb')} &= G_{cc'} - \frac{G_{cb}G_{bc'}}{G_{bb}} - \frac{G_{cb'}G_{b'c'}}{G_{b'b'}} + \frac{G_{cb}G_{bb'}G_{b'c'}}{G_{bb}G_{b'b'}} + O(|G_{bb'}|^2 + |G_{bc'}|^2 + |G_{cb'}|^2 + |G_{cc'}|^2) \\ &= G_{cc'} - \frac{G_{bc'}}{\sqrt{d-1}} - \frac{G_{cb'}}{\sqrt{d-1}} + \frac{G_{bb'}}{d-1} + \mathcal{E}, \end{aligned} \quad (12.19) \quad \{\text{e:remove_on}\}$$

where $|\mathcal{E}| \lesssim N^{-b}(|G_{bb'}| + |G_{bc'}| + |G_{cb'}| + |G_{cc'}|)$.

By plugging (12.19) into the first statement in (12.16), we get

$$\begin{aligned} \frac{1}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} (G_{cc'}^{(bb')})^2 &= \frac{1}{(Nd)^2} \sum_{\substack{b \sim c \\ b' \sim c'}} \left(G_{cc'} - \frac{G_{bc'}}{\sqrt{d-1}} - \frac{G_{cb'}}{\sqrt{d-1}} + \frac{G_{bb'}}{d-1} \right)^2 + O(N^{-b/2}\Phi) \\ &= \frac{1}{\mathcal{A}^2} \frac{\partial_z m(z)}{N} + O(N^{-b/2}\Phi). \end{aligned} \quad (12.20) \quad \{\text{e:square1}\}$$

where in the first statement we used the Ward identity (7.52) to `{e:square1}` error; and the second statement follows from repeated using the following relations

$$\frac{1}{\sqrt{d-1}} \sum_{x \sim u} G_{xv} = (HG)_{uv} = zG_{uv} + \delta_{uv} = 2G_{uv} + \delta_{uv} + O(N^{-g}|G_{uv}|), \quad (12.21) \quad \{\text{e:sum_neigh}\}$$

$$\frac{1}{N^2} \sum_{x,y} G_{xy}^2 = \frac{\text{Tr}[G^2]}{N^2} = \frac{\partial_z m(z)}{N} \quad (12.22) \quad \{\text{e:sum_neighb}\}$$

□

13 Error from local resampling

In this section we prove the estimates for the error terms arising from the local resampling used to bound (7.29). A key ingredient is the following *punctured-vertex* Ward bound. `{s:change_est}`

Proposition 13.1. *We take $z \in \mathbf{D}$, and recall the indicator functions*

$$I(\{o, i\}, \mathcal{G}) = A_{oi} \mathbf{1}(\mathcal{B}_{\mathfrak{R}}(o, \mathcal{G}) \text{ is a tree}) \quad (13.1)$$

from (7.3). Then the following holds for $i, j \sim o$ and $i \neq j$, `{lem:deleteda}`

$$\frac{1}{N} \sum_{o \in [N]} \mathbb{E}[I(\{i, o\}, \mathcal{G}) |G_{ij}^{(o)}|^2] \lesssim \mathbb{E}[N^e \Phi]. \quad (13.2) \quad \{\text{e:sameasdis}\}$$

As a consequence of Proposition 13.1, the following proposition states that during the local resampling, the errors $\mathcal{E}$ from Lemma 7.4 (after averaging) are negligible. Proposition 13.2 follows from Proposition 13.1

and Schur complement formula (2.4). The proofs of Proposition 13.1 and Proposition 13.2 will be given in Section 13.2.

We recall the resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$ around o from Section 5, and let

$$\mathcal{F} := \{i, o\}, \quad \mathcal{F}^+ := B_\ell(o, \mathcal{G}) \cup \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]} = B_{\ell+1}(o, \mathcal{G}) \cup \{(b_\alpha, c_\alpha)\}_{\alpha \in [\mu]} = (\mathbf{i}^+, E^+), \quad (13.3) \quad \{\mathbf{e}:\mathbf{cF_copy}\}$$

which contains all the switching edges, see Figure 4. We also recall the following indicator function from (8.6) $\{\mathbf{e}:\mathbf{cF_copy}\}$

$$I(\mathcal{F}, \mathcal{G}) := A_{io} \prod_{x \in \mathcal{B}_\ell(o; \mathcal{G})} \mathbf{1}(\mathcal{B}_\mathcal{R}(x; \mathcal{G}) \text{ is a tree}) \quad (13.4) \quad \{\mathbf{e}:\mathbf{defI_copy}\}$$

$$I(\mathcal{F}^+, \mathcal{G}) := \prod_{\{x, y\} \in E^+} A_{xy} \prod_{\substack{c \in \{o, c_\alpha, \dots, c_\mu\} \\ x \in \mathcal{B}_\ell(c; \mathcal{G})}} \mathbf{1}(\mathcal{B}_\mathcal{R}(x; \mathcal{G}) \text{ is a tree}) \prod_{c \neq c' \in \{o, c_1, \dots, c_\mu\}} \mathbf{1}(\text{dist}_\mathcal{G}(c, c') \geq 3\mathfrak{R}). \quad (13.5) \quad \{\mathbf{e}:\mathbf{defI_copy}\}$$

$\{\mathbf{lem}:\mathbf{task2}\}$

Proposition 13.2. We take $z \in \mathbf{D}$, and denote the resampling data $\mathbf{S} = \{(l_\alpha, a_\alpha), (b_\alpha, c_\alpha)\}_{\alpha \in [\mu]}$, the following holds

$$\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) | \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}] \lesssim (d-1)^\ell \mathbb{E}[N^\circ \Phi]. \quad (13.6) \quad \{\mathbf{eq}:\mathbf{task2}\}$$

13.1 Schur complement formula revisit

$\{\mathbf{s}:\mathbf{Schur_revi}\}$

Adopt the notations as in the proof of Lemma 7.4, and condition on that $I(\mathcal{F}^+, \mathcal{G}) = 1$, so the switching edges have tree neighborhood and are far away from each other. Then the normalized adjacency matrix $\tilde{H}^{(\mathbb{T})}$ of $\tilde{\mathcal{G}}^{(\mathbb{T})}$ is in the block form

$$\tilde{H}^{(\mathbb{T})} = \begin{bmatrix} \tilde{H}_{\mathbb{W}}^{(\mathbb{T})} & \tilde{B}^\top \\ \tilde{B} & \tilde{H}_{\mathbb{W}^c}^{(\mathbb{T})} \end{bmatrix}.$$

We also denote the Green's function of $\mathcal{G}^{(\mathbb{T})}$ and $\tilde{\mathcal{G}}^{(\mathbb{T})}$ as $G^{(\mathbb{T})}$ and $\tilde{G}^{(\mathbb{T})}$ respectively.

In this section, we investigate the error from replacing $\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}$ with $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$. Then $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ can be obtained from $\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}$ through the following steps. First, we remove $\mathbb{W} = \{b_1, b_2, \dots, b_\mu\}$, which gives $G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{W})}$; we then add $\mathbb{W} \setminus \{b_\alpha, b_\beta\}$ back, which gives $G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{b}_\alpha b_\beta)}$; finally we add $\mathbb{T}$ back, which gives $G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$. The errors from these replacements are explicit, thanks to the Schur complement formulas (2.4): $\{\mathbf{eq}:\mathbf{task2}\}$

$$\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{W})} = (G^{(\mathbb{T}\mathbb{W})} \tilde{B} \tilde{G}^{(\mathbb{T})} |_{\mathbb{W}} \tilde{B}^\top G^{(\mathbb{T}\mathbb{W})})_{c_\alpha c_\beta}, \quad (13.7) \quad \{\mathbf{e}:\mathbf{removeW}\}$$

$$G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{W})} - G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{b}_\alpha b_\beta)} = -(G^{(\mathbb{T}\mathbb{W})} (G^{(\mathbb{T}\mathbb{b}_\alpha b_\beta)} |_{\mathbb{W} \setminus \{b_\alpha, b_\beta\}})^{-1} G^{(\mathbb{T}\mathbb{b}_\alpha b_\beta)})_{c_\alpha c_\beta}, \quad (13.8) \quad \{\mathbf{e}:\mathbf{addW}\}$$

$$G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} - G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{b}_\alpha b_\beta)} = (G^{(b_\alpha b_\beta)} (G^{(b_\alpha b_\beta)} |_{\mathbb{T}})^{-1} G^{(b_\alpha b_\beta)})_{c_\alpha c_\beta}. \quad (13.9) \quad \{\mathbf{e}:\mathbf{schur_rem}\}$$

The following lemma provide leading order terms for the replacement errors associated with the equations (13.7), (13.8) and (13.9): $\{\mathbf{1}:\mathbf{diffG1}\}$

Lemma 13.3. Fix $z \in \mathbf{D}$ and recall $\mathcal{F}, \mathcal{F}^+$ from (13.3). We denote $\mathcal{T} = \mathcal{B}_\ell(o, \mathcal{G})$ with vertex set $\mathbb{T}$. We assume that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$. For any indices $\alpha, \beta \in [\mu]$, the following holds:

1. The difference $\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{W})}$, is given by

$$\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} - G_{c_\alpha c_\beta}^{(\mathbb{T}\mathbb{W})} = \frac{m_d(z)}{d-1} \sum_{\gamma \in [\mu]} \left(\sum_{x \in \mathcal{N}_\gamma} G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})} \right) \left(\sum_{x \in \mathcal{N}_\gamma} G_{c_\beta x}^{(\mathbb{T}\mathbb{W})} \right) + \mathcal{E}, \quad (13.10) \quad \{\mathbf{e}:\mathbf{diffG1}\}$$

where $\mathcal{N}_\gamma = \{x \neq c_\gamma : x \sim b_\gamma \text{ in } \mathcal{G}\} \cup \{a_\gamma\}$, which enumerates the adjacent vertices of b_γ in $\tilde{\mathcal{G}}$, and

$$|\mathcal{E}| \lesssim N^{-\mathfrak{b}/2} \sum_{\gamma \in \llbracket \mu \rrbracket} \sum_{x \in \mathcal{N}_\gamma} (|G_{c_\alpha x}^{(\text{TW})}|^2 + |G_{c_\beta x}^{(\text{TW})}|^2) + N^{-\mathfrak{b}} \Phi.$$

Moreover,

$$|\tilde{G}_{c_\alpha c_\beta}^{(\text{T})} - G_{c_\alpha c_\beta}^{(\text{TW})}| \lesssim \sum_{\gamma \in \llbracket \mu \rrbracket} \sum_{x \in \mathcal{N}_\gamma} (|G_{c_\alpha x}^{(\text{TW})}|^2 + |G_{c_\beta x}^{(\text{TW})}|^2) + N^{-\mathfrak{b}} \Phi. \quad (13.11) \quad \{\mathbf{e}:\text{total_diff}\}$$

2. The difference $G_{c_\alpha c_\beta}^{(\text{TW})} - G_{c_\alpha c_\beta}^{(\text{T}b_\alpha b_\beta)}$ is given by

$$G_{c_\alpha c_\beta}^{(\text{TW})} - G_{c_\alpha c_\beta}^{(\text{T}b_\alpha b_\beta)} = -\frac{1}{m_d(z)} \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha b_\beta)} G_{b_\gamma c_\beta}^{(\text{T}b_\alpha b_\beta)} + \mathcal{E}, \quad (13.12) \quad \{\mathbf{e}:\text{diffG2}\}$$

where

$$|\mathcal{E}| \lesssim N^{-\mathfrak{b}/2} \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} |G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha b_\beta)}|^2 + |G_{c_\beta b_\gamma}^{(\text{T}b_\alpha b_\beta)}|^2.$$

Moreover,

$$|G_{c_\alpha c_\beta}^{(\text{TW})} - G_{c_\alpha c_\beta}^{(\text{T}b_\alpha b_\beta)}| \lesssim \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} (|G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha b_\beta)}|^2 + |G_{c_\beta b_\gamma}^{(\text{T}b_\alpha b_\beta)}|^2). \quad (13.13) \quad \{\mathbf{e}:\text{total_diff}\}$$

3. The difference $G_{c_\alpha c_\beta}^{(\text{T}b_\alpha b_\beta)} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}$ is given by

$$G_{c_\alpha c_\beta}^{(\text{T}b_\alpha b_\beta)} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} = - \sum_{\substack{\text{dist}(x, o) = \ell \\ x \sim x' \in \mathbb{T}}} \left(\frac{1}{\sqrt{d-1}} G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{x' c_\beta}^{(b_\alpha b_\beta)} + \frac{1}{m_{\text{sc}}(z)} G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{x c_\beta}^{(b_\alpha b_\beta)} \right) + \mathcal{E}, \quad (13.14) \quad \{\mathbf{e}:\text{diffG3}\}$$

where

$$|\mathcal{E}| \lesssim N^{-\mathfrak{b}/2} \sum_{x \in \mathbb{T}} (|G_{c_\alpha x}^{(b_\alpha b_\beta)}|^2 + |G_{c_\beta x}^{(b_\alpha b_\beta)}|^2).$$

Moreover,

$$|G_{c_\alpha c_\beta}^{(\text{T}b_\alpha b_\beta)} - G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| \lesssim \sum_{x \in \mathbb{T}} (|G_{c_\alpha x}^{(b_\alpha b_\beta)}|^2 + |G_{c_\beta x}^{(b_\alpha b_\beta)}|^2). \quad (13.15) \quad \{\mathbf{e}:\text{total_diff}\}$$

4. The following holds

$$\begin{aligned} |G_{c_\alpha a_\beta}^{(\text{TW})} - G_{c_\alpha a_\beta}^{(b_\alpha l_\beta)}| &\lesssim \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha\}} (|G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha)}|^2 + |G_{a_\beta b_\gamma}^{(\text{T}b_\alpha)}|^2) + \sum_{y \in \mathbb{T} \setminus \{l_\beta\}} (|G_{c_\alpha y}^{(b_\alpha l_\beta)}|^2 + |G_{a_\beta y}^{(b_\alpha l_\beta)}|^2), \\ |G_{c_\alpha x}^{(\text{TW})} - G_{c_\alpha x}^{(b_\alpha b_\beta)}| &\lesssim \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} (|G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha b_\beta)}|^2 + |G_{b_\gamma c_\beta}^{(\text{T}b_\alpha b_\beta)}|^2) + \sum_{y \in \mathbb{T}} (|G_{c_\alpha y}^{(b_\alpha b_\beta)}|^2 + |G_{xy}^{(b_\alpha b_\beta)}|^2), \text{ for } x \sim b_\beta, \\ |G_{c_\alpha b_\theta}^{(\text{T}b_\alpha b_\beta)} - G_{c_\alpha b_\theta}^{(b_\alpha)}| &\lesssim (|G_{c_\alpha b_\beta}^{(\text{T}b_\alpha)}|^2 + |G_{b_\theta b_\beta}^{(\text{T}b_\alpha)}|^2) + \sum_{y \in \mathbb{T}} (|G_{c_\alpha y}^{(b_\alpha)}|^2 + |G_{b_\theta y}^{(b_\alpha)}|^2), \text{ for } \theta \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}. \end{aligned} \quad (13.16) \quad \{\mathbf{e}:\text{Greplace}\}$$

Proof of Lemma 13.3. We can decompose the right-hand side of (13.7) is $\{\mathbf{e}:\text{Greplace}\}$

$$I := \frac{1}{d-1} \sum_{\gamma, \gamma' \in \llbracket \mu \rrbracket} \sum_{\substack{x \in \mathcal{N}_\gamma \\ y \in \mathcal{N}_{\gamma'}}} G_{c_\alpha x}^{(\text{TW})} \tilde{G}_{b_\gamma b_{\gamma'}}^{(\text{T})} G_{y c_\beta}^{(\text{TW})},$$

where $\mathcal{N}_\gamma = \{x \neq c_\gamma : x \sim b_\gamma \text{ in } \mathcal{G}\} \cup \{a_\gamma\}$.

Since $\tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$, (5.7) gives that for $\gamma \neq \gamma'$, $|\tilde{G}_{b_\gamma b_{\gamma'}}^{(\mathbb{T})}| \lesssim 1$ and $|\tilde{G}_{b_\gamma b_{\gamma'}}^{(\mathbb{T})}| \lesssim N^{-\mathfrak{b}}$. The leading order term of I is then for pairs $\gamma = \gamma'$, giving

$$I = \frac{m_d(z)}{d-1} \sum_{\gamma \in \llbracket \mu \rrbracket} \left(\sum_{x \in \mathcal{N}_\gamma} G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})} \right) \left(\sum_{x \in \mathcal{N}_\gamma} G_{c_\beta x}^{(\mathbb{T}\mathbb{W})} \right) + O \left(N^{-\mathfrak{b}} \sum_{\gamma, \gamma' \in \llbracket \mu \rrbracket} \sum_{\substack{x \in \mathcal{N}_\gamma \\ y \in \mathcal{N}_{\gamma'}}} |G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})}| |G_{y c_\beta}^{(\mathbb{T}\mathbb{W})}| \right). \quad (13.17) \quad \{\mathfrak{e}: \text{II_approx}\}$$

The claim (13.10) follows from (13.17).

To prove (13.12), we start with (13.8). Consider

$$(G^{(\mathbb{T}b_\alpha b_\beta)} (G^{(\mathbb{T}b_\alpha b_\beta)}|_{\mathbb{W} \setminus \{b_\alpha, b_\beta\}})^{-1} G^{(\mathbb{T}b_\alpha b_\beta)})_{c_\alpha c_\beta} = \sum_{\gamma, \gamma' \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)} (G^{(\mathbb{T}b_\alpha b_\beta)}|_{\mathbb{W} \setminus \{b_\alpha, b_\beta\}})^{-1}_{b_\gamma b_{\gamma'}} G_{b_{\gamma'} c_\beta}^{(\mathbb{T}b_\alpha b_\beta)}. \quad (13.18) \quad \{\mathfrak{e}: \text{defcElbet}\}$$

The leading order term in (13.18) is (given by (13.18)) with $\gamma = \gamma'$,

$$(13.18) = \frac{1}{m_d(z)} \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)} G_{b_\gamma c_\beta}^{(\mathbb{T}b_\alpha b_\beta)} + \mathcal{E}, \quad (13.19) \quad \{\mathfrak{e}: \text{IVerror}\}$$

$$\mathcal{E} := \sum_{\gamma, \gamma' \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)} ((G^{(\mathbb{T}b_\alpha b_\beta)}|_{\mathbb{W} \setminus \{b_\alpha, b_\beta\}})^{-1}_{b_\gamma b_{\gamma'}} - \delta_{\gamma\gamma'} / m_d(z)) G_{b_{\gamma'} c_\beta}^{(\mathbb{T}b_\alpha b_\beta)}.$$

Thanks to (3.26), $|G_{b_\gamma b_{\gamma'}}^{(\mathbb{T}b_\alpha b_\beta)}| m_d(z) \delta_{\gamma\gamma'} \lesssim N^{-\mathfrak{b}}$. Thus $|(G^{(\mathbb{T}b_\alpha b_\beta)}|_{\mathbb{W} \setminus \{b_\alpha, b_\beta\}})^{-1}_{b_\gamma b_{\gamma'}} - \delta_{\gamma\gamma'} / m_d(z)| \lesssim N^{-3\mathfrak{b}/4}$, and the error $\mathcal{E}$ in (13.19) is bounded as

$$|\mathcal{E}| \lesssim N^{-3\mathfrak{b}/4} \sum_{\gamma, \gamma' \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} |G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}| |G_{b_{\gamma'} c_\beta}^{(\mathbb{T}b_\alpha b_\beta)}| \lesssim N^{-\mathfrak{b}/2} \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} |G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2 + |G_{c_\beta b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2. \quad (13.20) \quad \{\mathfrak{e}: \text{IVerror2}\}$$

The claim (13.12) follows from combining (13.19) and (13.20).

To prove (13.14), we can rewrite the right-hand side of (13.9) explicitly as

$$-(G^{(b_\alpha b_\beta)} (G^{(b_\alpha b_\beta)}|_{\mathbb{T}})^{-1} G^{(b_\alpha b_\beta)})_{c_\alpha c_\beta} = - \sum_{x, y \in \mathbb{T}} G_{c_\alpha x}^{(b_\alpha b_\beta)} (G^{(b_\alpha b_\beta)}|_{\mathbb{T}})^{-1}_{xy} G_{y c_\beta}^{(b_\alpha b_\beta)}. \quad (13.21) \quad \{\mathfrak{e}: \text{schur_rem}\}$$

Since $I(\mathcal{F}^+, \mathcal{G}) = 1$, by (3.26), for $x, y \in \mathbb{T}$, $|(G^{(b_\alpha b_\beta)}|_{\mathbb{T}})^{-1}_{xy} - (H_{\mathbb{T}} - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{xy}^{-1}| \lesssim N^{-\mathfrak{b}}$, where $\mathbb{I}_{xy}^\partial = \mathbf{1}(\text{dist}_{\mathcal{G}}(o, x) = \ell) \delta_{xy}$. Thus we have

$$|(G^{(b_\alpha b_\beta)}|_{\mathbb{T}})^{-1}_{xy} - (H_{\mathbb{T}} - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{xy}| \lesssim N^{-3\mathfrak{b}/4}, \text{ for } x, y \in \mathbb{T},$$

and

$$(13.21) = \sum_{x, y \in \mathbb{T}} (H_{\mathbb{T}} - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{xy} G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{y c_\beta}^{(b_\alpha b_\beta)} + O \left(N^{-3\mathfrak{b}/4} \sum_{x, y \in \mathbb{T}} |G_{c_\alpha x}^{(b_\alpha b_\beta)}| |G_{y c_\beta}^{(b_\alpha b_\beta)}| \right). \quad (13.22) \quad \{\mathfrak{e}: \text{diffG4}\}$$

For the summation over $x, y \in \mathbb{T}$ in (13.22), if $\text{dist}_{\mathcal{G}}(x, o) < \ell$, then $\mathbb{I}_{xy}^\partial = 0$, and we have

$$\sum_{y \in \mathbb{T}} (H_{\mathbb{T}} - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{xy} G_{y c_\beta}^{(b_\alpha b_\beta)} = \sum_{y \in \mathbb{T}} (H_{\mathbb{T}} - z)_{xy} G_{y c_\beta}^{(b_\alpha b_\beta)} = 0, \quad (13.23) \quad \{\mathfrak{e}: \text{HGexp}\}$$

where for the last equality, we used that by the definition of the Green's function, the first term on the second line is $\delta_{x c_\beta} = 0$. Thus by plugging (13.23) into (13.22), it follows that

$$\sum_{\substack{\text{dist}(x, o) < \ell \\ y \in \mathbb{T}}} G_{c_\alpha x}^{(b_\alpha b_\beta)} (H_{\mathbb{T}} - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{xy} G_{y c_\beta}^{(b_\alpha b_\beta)} = 0. \quad (13.24) \quad \{\mathfrak{e}: \text{interior}\}$$

If $x \in \mathbb{T}$ and $\text{dist}(x, o) = \ell$, we denote the parent node of x as x' . We then have, by the self-consistent equation of m_{sc} ,

$$\begin{aligned} & \sum_{y \in \mathbb{T}} G_{c_\alpha x}^{(b_\alpha b_\beta)} (H - z - m_{\text{sc}}(z) \mathbb{I}^\partial)_{xy} G_{yc_\beta}^{(b_\alpha b_\beta)} \\ &= \frac{1}{\sqrt{d-1}} G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{x'c_\beta}^{(b_\alpha b_\beta)} - (z + m_{\text{sc}}(z)) G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{xc_\beta}^{(b_\alpha b_\beta)} \\ &= \frac{1}{\sqrt{d-1}} G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{x'c_\beta}^{(b_\alpha b_\beta)} + \frac{1}{m_{\text{sc}}(z)} G_{c_\alpha x}^{(b_\alpha b_\beta)} G_{xc_\beta}^{(b_\alpha b_\beta)}. \end{aligned} \quad (13.25) \quad \{\mathbf{e:boundary}\}$$

The claim (13.14) follows from plugging (13.24) and (13.25) into (13.22).

The claims (13.16) follow from the same arguments as in (13.12) and (13.14), so we omit the proof. $\square$

13.2 Proof of Proposition 13.1 and Proposition 13.2

Proof of Proposition 13.1. We can replace the indicator function I_o by $I(\mathcal{F}, \mathcal{G})$ by the same argument for Claim 7.1

$$\begin{aligned} & \frac{1}{N} \sum_{o \in [N]} \mathbb{E}[|G_{ij}^{(o)}|^2] = \frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) |G_{ij}^{(o)}|^2 \right] + O(N^{-1+\epsilon}) \\ & \lesssim \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) |\tilde{G}_{ij}^{(o)}|^2 \right] + O(N^{-\mathfrak{b}} \mathbb{E}[\Phi]). \end{aligned} \quad (13.26) \quad \{\mathbf{e:switching}\}$$

We are now left to write the Green's function of the switched graph in terms of the original graph. Let $\mathcal{T} = B_\ell(o, \mathcal{G})$ and $L := P(\mathcal{T}, z, m_{\text{sc}}(z))$. Here L is consistent with the local Green's function (as defined in (8.13)) ~~de:switching~~ use the same symbols to represent them. We notice that since i, j are distinct neighbors of o , so i, j are in different connected components of $\mathcal{T}^{(o)}$. Thus $L_{ij}^{(o)} = 0$, and $\tilde{G}_{ij}^{(o)} = \tilde{G}_{ij}^{(o)} - L_{ij}^{(o)}$. We will use the same argument as in the proof of Lemma 7.4. In the rest, we condition on that $\mathcal{G}, \tilde{\mathcal{G}} \in \Omega$ and $I(\mathcal{F}^+, \mathcal{G}) = 1$. Then by the same argument as for (7.19), we have

$$\tilde{G}_{ij}^{(o)} = \left(L^{(o)} \sum_{k=1}^{\mathfrak{p}} \left((\tilde{B}^\top (\tilde{G}^{(\mathbb{T})} - m_{\text{sc}}(z)) \tilde{B}) L^{(o)} \right)^k \right)_{ij} + O(N^{-2}). \quad (13.27) \quad \{\mathbf{e:GooY2}\}$$

For any $1 \leq k \leq \mathfrak{p}$, the k -th term in (13.27) is ~~an $O(N^{-2})$~~ weighted sum of terms of the following form

$$\sum_{x_1, x_2, \dots, x_{2k} \in \mathbb{T}} L_{ix_1}^{(o)} V_{x_1 x_2} L_{x_2 x_3}^{(o)} V_{x_3 x_4} L_{x_4 x_5}^{(o)} \cdots V_{x_{2k-1} x_{2k}} L_{x_{2k} j}^{(o)}. \quad (13.28) \quad \{\mathbf{e:PUP2}\}$$

Here $V = (\tilde{B}^\top (\tilde{G}^{(\mathbb{T})} - m_{\text{sc}}(z)) \tilde{B})$, and (5.7) gives ~~resp. (13.28)~~ $|V_{xy}| \lesssim N^{-\mathfrak{b}}$ for any $x, y \in \mathbb{T}$.

We recall that i, j are in different connected components of $\mathcal{T}^{(o)}$. For the sequence of indices $x_0 := i, x_1, x_2, \dots, x_{2k}, x_{2k+1} := j$, there exists some pair of consecutively listed vertices that are in different connected components of $\mathcal{T}^{(o)}$. If for some $0 \leq m \leq k$, x_{2m}, x_{2m+1} are in different connected components of $\mathcal{T}^{(o)}$, then $L_{x_{2m} x_{2m+1}}^{(o)} = 0$ and (13.28) vanishes. Thus we only need to consider the case that for some $1 \leq m \leq k$, x_{2m-1}, x_{2m} are in different connected components of $\mathcal{T}^{(o)}$. In this case,

$$V_{x_{2m-1}, x_{2m}} = (\tilde{B}^\top (\tilde{G}^{(\mathbb{T})} - m_{\text{sc}}(z)) \tilde{B})_{x_{2m-1}, x_{2m}} = \sum_{\alpha, \beta: l_\alpha = x_{2m-1}, l_\beta = x_{2m}} \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}. \quad (13.29) \quad \{\mathbf{e:Vxm}\}$$

If $k = 1$, then $m = 1$ and we can compute (13.28) using (13.29) as

$$\frac{1}{d-1} \sum_{\alpha \neq \beta \in [\mu]} L_{il_\alpha}^{(o)} \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} L_{l_\beta j}^{(o)} = \frac{m_{\text{sc}}^{2\ell}(z)}{(d-1)^\ell} \sum_{\substack{\text{dist}_{\mathcal{T}}(i, l_\alpha) = \ell-1 \\ \text{dist}_{\mathcal{T}}(j, l_\beta) = \ell-1}} \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}, \quad (13.30) \quad \{\mathbf{e:k=1Pxx}\}$$

where we used (7.17), and $\frac{1}{N} \leq O(1)$. For $k \geq 2$, we can bound (13.28) as

$$\begin{aligned} (13.28) &\lesssim \sum_{\alpha \neq \beta} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}| \sum_{\substack{x_1, x_2, \dots, x_{2m-2} \in \mathbb{T} \\ x_{2m+1}, x_{2m+2}, \dots, x_{2k} \in \mathbb{T}}} |L_{ix_1}^{(o)}| \cdots |L_{x_{2m-2}l_\alpha}^{(o)}| |L_{l_\beta x_{2m+1}}^{(o)}| \cdots |L_{x_{2k}j}^{(o)}| \\ &\lesssim N^{-(k-1)b} \sum_{\alpha \neq \beta} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}| (d-1)^\ell (\ell(d-1)^\ell)^{k-1} \lesssim \frac{1}{(d-1)^{7(k-1)b/8}} \sum_{\alpha \neq \beta} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}|, \end{aligned} \quad (13.31) \quad \{\mathbf{e}:\text{sumx1234}\}$$

where the first statement follows from (13.29) and $\frac{1}{N} \leq O(1)$; the second statement follows from (7.17); in the third statement we used $N^b \geq (d-1)^{20\ell}$.

The estimates (13.30) and (13.31) together lead to the following estimate for (13.27)

$$(13.27) = \frac{m_{\mathcal{F}}^{2\ell}(z)}{(d-1)^\ell} \sum_{\substack{\alpha \in A_i \\ \beta \in A_j}} \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} + O\left(\frac{1}{N^{7b/8}} \sum_{\alpha \neq \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}| + N^{-2}\right), \quad (13.32) \quad \{\mathbf{e}:\text{firstbound}\}$$

and by plugging (13.32) back into (13.26) we conclude that

$$\begin{aligned} &\frac{1}{Z_{\mathcal{F}}} \sum_{\mathbf{i}} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}, \mathcal{G}) |G_{ij}^{(o)}|^2 \right] \lesssim J_1 + J_2 + O(N^{-b} \mathbb{E}[\Phi]) \\ J_1 &:= \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) \left| \frac{1}{(d-1)^\ell} \sum_{\alpha \neq \beta \in \llbracket \mu \rrbracket} \tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} \right|^2 \right] \\ |J_2| &\lesssim \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) N^{-3b/4} \left(\sum_{\alpha \neq \beta \in \llbracket \mu \rrbracket} |\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}|^2 + \Phi \right) \right]. \end{aligned} \quad (13.33) \quad \{\mathbf{e}:\text{IIGU2}\}$$

Next, we estimate J_1 and J_2 as in (13.33). We need to express $\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})}$ in terms of the Green's function of the graph $\mathcal{G}$. In this process, any term that can be bounded by $O(N^{-o(1)}\Phi)$ is considered negligible, since it contributes to an error $\mathbb{E}[\mathbf{1}(\mathcal{G} \in \Omega) N^{-o(1)}\Phi] = O(N^{-o(1)}\mathbb{E}[\Phi])$.

Thanks to Lemma 13.3 (combining (13.11), (13.13) and (13.15)), we have

$$\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} = G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{E}_{\alpha\beta}, \quad (13.34) \quad \{\mathbf{e}:\text{replace}\}$$

where

$$\begin{aligned} |\mathcal{E}_{\alpha\beta}| &\lesssim \sum_{\gamma \in \llbracket \mu \rrbracket, x \in \mathcal{N}_\gamma} (|G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})}|^2 + |G_{c_\beta x}^{(\mathbb{T}\mathbb{W})}|^2) + \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} (|G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2 + |G_{c_\beta b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2) \\ &\quad + \sum_{x \in \mathbb{T}} (|G_{c_\alpha x}^{(b_\alpha b_\beta)}|^2 + |G_{c_\beta x}^{(b_\alpha b_\beta)}|^2) + N^{-b}\Phi, \end{aligned} \quad (13.35) \quad \{\mathbf{e}:\text{cEbound}\}$$

and $N_\gamma = \{x \neq c_\gamma : x \sim b_\gamma\} \cup \{a_\gamma\}$.

In the following we show that for $\mathcal{G} \in \Omega$, and $\alpha \neq \beta, \alpha' \neq \beta'$

$$\begin{aligned} &\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) (|\mathcal{E}_{\alpha\beta}| |G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})}| + |G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}| |\mathcal{E}_{\alpha'\beta'}| + |\mathcal{E}_{\alpha\beta}|^2) \\ &\lesssim N^{-b} \sum_{\substack{\gamma \in \{\alpha, \beta, \alpha', \beta'\} \\ x \sim b_\gamma, x \neq c_\gamma}} \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) |G_{c_\gamma x}^{(b_\gamma)}|^2 + \frac{\Phi}{N^{3b/4}}, \end{aligned} \quad (13.36) \quad \{\mathbf{e}:\text{smallterm}\}$$

and thus

$$\begin{aligned} & \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G})(G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{E}_{\alpha\beta})(\overline{G}_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})} + \overline{\mathcal{E}}_{\alpha'\beta'}) \\ &= \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} \overline{G}_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})} + o \left(\sum_{\substack{\gamma \in \{\alpha, \beta, \alpha', \beta'\} \\ x \sim b_\gamma, x \neq c_\gamma}} \frac{N^{-b}}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) |G_{c_\gamma x}^{(b_\gamma)}|^2 + \frac{\Phi}{N^{3b/4}} \right). \end{aligned} \quad (13.37) \quad \{\mathbf{e:GGGxy}\}$$

To prove (13.36), we can bound the left-hand side of (13.36), after which each term contains three Green's function entries as factors. We can bound one of them by N^{-b} using (3.26), and the remaining two can be bounded by terms in the form $\{|G_{c_\gamma x}^{(b_\gamma)}|^2\}$ with $\gamma \in \{\alpha, \beta, \alpha', \beta'\}, x \sim b_\gamma, x \neq c_\gamma$ or can be bounded by $N^o \Phi$ using (3.27). In the following we estimate the following term from (13.36), and the other terms can be bounded in the same way, so we omit arguments about them.

$$\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) \sum_{\gamma \in [\mu], x \in \mathcal{N}_\gamma} |G_{c_\alpha x}^{(\text{TW})}|^2 |G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})}| \lesssim \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) \sum_{\gamma \in [\mu], x \in \mathcal{N}_\gamma} N^{-b} |G_{c_\alpha x}^{(\text{TW})}|^2, \quad (13.38) \quad \{\mathbf{e:yizhong}\}$$

where we bound $|G_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})}| \lesssim N^{-b}$ by (3.26). If $\gamma \neq \alpha$, thanks to (3.27), we have (ignore the indicator and the averaging over embeddings)

$$(13.38) \lesssim \frac{1}{Nd} \sum_{\substack{\gamma \in [\mu] \setminus \{\alpha\} \\ x \in \mathcal{N}_\gamma}} N^{-b} \sum_{b_\alpha \sim c_\alpha} |G_{c_\alpha x}^{(\text{TW})}|^2 \lesssim N^{-b} N^o \Phi \lesssim N^{-3b/4} \Phi. \quad (13.39) \quad \{\mathbf{e:yizhong2}\}$$

Thus we can reduce (13.38) to $\{\mathbf{e:yizhong2}\} = \alpha$

$$\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) N^{-b} \left(|G_{c_\alpha a_\alpha}^{(\text{TW})}|^2 + \sum_{\substack{x \sim b_\alpha \\ x \neq c_\alpha}} |G_{c_\alpha x}^{(\text{TW})}|^2 \right). \quad (13.40) \quad \{\mathbf{e:GtGG}\}$$

The terms involving $|G_{c_\alpha a_\alpha}^{(\text{TW})}|^2$ can be bounded by the same way as in (13.39). Next, we show that we can replace $G_{c_\alpha x}^{(\text{TW})}$ in (13.40) by $G_{c_\alpha x}^{(b_\alpha)}$.

$$\begin{aligned} (13.40) &= \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) N^{-b} \sum_{\substack{x \sim b_\alpha \\ x \neq c_\alpha}} |G_{c_\alpha x}^{(b_\alpha)}|^2 + \mathcal{E}, \\ |\mathcal{E}| &\lesssim \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) N^{-b} \sum_{\substack{x \sim b_\alpha \\ x \neq c_\alpha}} |G_{c_\alpha x}^{(\text{TW})} - G_{c_\alpha x}^{(b_\alpha)}| + o(N^{-3b/4} \Phi), \end{aligned} \quad (13.41) \quad \{\mathbf{e:yizhong3}\}$$

where for the bound of $\mathcal{E}$, we used that $|G_{c_\alpha x}^{(\text{TW})}|, |G_{c_\alpha x}^{(b_\alpha)}| \lesssim 1$ from (3.26). $\{\mathbf{e:yizhong3}\}$

Thanks to (13.16), we can bound the difference $|G_{c_\alpha x}^{(\text{TW})} - G_{c_\alpha x}^{(b_\alpha)}|$ by

$$\left| G_{c_\alpha x}^{(\text{TW})} - G_{c_\alpha x}^{(b_\alpha)} \right| \leq \sum_{\gamma \in [\mu] \setminus \{\alpha\}} (|G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha)}|^2 + |G_{x b_\gamma}^{(\text{T}b_\alpha)}|^2) + \sum_{y \in \mathbb{T}} (|G_{c_\alpha y}^{(b_\alpha)}|^2 + |G_{xy}^{(b_\alpha)}|^2). \quad (13.42) \quad \{\mathbf{e:yizhong4}\}$$

By plugging (13.42) into (13.41) by the same argument as in (13.39), we can bound $\mathcal{E}$ in (13.41) as

$$\frac{N^{-b}}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) \left(\sum_{\gamma \in [\mu] \setminus \{\alpha\}} (|G_{c_\alpha b_\gamma}^{(\text{T}b_\alpha)}|^2 + |G_{x b_\gamma}^{(\text{T}b_\alpha)}|^2) + \sum_{y \in \mathbb{T}} (|G_{c_\alpha y}^{(b_\alpha)}|^2 + |G_{xy}^{(b_\alpha)}|^2) \right) \lesssim N^{-3b/4} \Phi. \quad (13.43) \quad \{\mathbf{e:yizhong5}\}$$

The claim (13.36) follows from plugging (13.39), (13.41) and (13.43) into (13.38).

For the first term on the right-hand side of (13.37), we recall that $\alpha \neq \beta$ and $\alpha' \neq \beta'$. Then either $\{\alpha, \beta\} = \{\alpha', \beta'\}$, or some indices, say α, α' , only appears once (namely, $\alpha \neq \alpha', \beta'$ and $\alpha' \neq \alpha, \beta$). Then we can sum over (b_α, c_α) and $(b_{\alpha'}, c_{\alpha'})$ separately (ignoring the indicator function)

$$\begin{aligned} & \frac{1}{(Nd)^2} \sum_{c_\alpha \sim b_\alpha} \sum_{c_{\alpha'} \sim b_{\alpha'}} G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} \overline{G}_{c_{\alpha'} c_{\beta'}}^{(b_{\alpha'} b_{\beta'})} \\ & \lesssim \frac{1}{(Nd)^2} \sum_{c_\alpha \sim b_\alpha} \sum_{c_{\alpha'} \sim b_{\alpha'}} N^{-b/2} (|G_{b_\alpha c_\beta}^{(b_\beta)}| + \Phi) N^{-b/2} (|G_{b_{\alpha'} c_{\beta'}}^{(b_{\beta'})}| + \Phi) \\ & \lesssim \frac{N^{-b}}{(Nd)^2} \sum_{c_\alpha \sim b_\alpha} \sum_{c_{\alpha'} \sim b_{\alpha'}} (|G_{b_\alpha c_\beta}^{(b_\beta)}|^2 + |G_{b_{\alpha'} c_{\beta'}}^{(b_{\beta'})}|^2 + \Phi) \lesssim N^{-b} \Phi, \end{aligned} \quad (13.44) \quad \{\mathbf{e:fbound1}\}$$

where in the first statement we used (11.28); the second statement follows from Cauchy-Schwarz inequality; and the last statement follows from the Ward-identity bound (3.27). Otherwise if $\{\alpha, \beta\} = \{\alpha', \beta'\}$, (7.51) gives

$$\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) |G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}|^2 \lesssim N^0 \Phi. \quad (13.45) \quad \{\mathbf{e:fbound2}\}$$

We recall that J_1 in (13.33) is obtained by averaging (13.37) over $\alpha \neq \beta \in [\mu]$ and $\alpha' \neq \beta' \in [\mu]$. By substituting (13.37), (13.44), and (13.45), we conclude that:

$$\begin{aligned} J_1 &= \sum_{\alpha \neq \beta \in [\mu]} \frac{1}{(d-1)^{2\ell} Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) |G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)}|^2 \right] \\ &+ \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) \left(\frac{(d-1)^{2\ell}}{N^b} \sum_{\alpha \in [\mu]} \sum_{x \sim b_\alpha, x \neq a_\alpha} |G_{c_\alpha x}^{(b_\alpha)}|^2 + \frac{\Phi}{N^{b/2}} \right) \right], \\ &\lesssim \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) \left(\frac{1}{N^{3b/4}} \sum_{x \sim b_\alpha, x \neq a_\alpha} |G_{c_\alpha x}^{(b_\alpha)}|^2 + N^0 \Phi \right) \right]. \end{aligned} \quad (13.46) \quad \{\mathbf{e:J1bound}\}$$

where in the last statement we used (13.45) and the permutation invariance of the vertices, so that the expectation does not depend on α . By the same argument we can also bound J_2 in (13.33) as,

$$J_2 \lesssim \frac{1}{N^{3b/4} Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) \left(\frac{1}{N^{3b/4}} \sum_{x \sim b_\alpha, x \neq a_\alpha} |G_{c_\alpha x}^{(b_\alpha)}|^2 + (d-1)^{2\ell} N^0 \Phi \right) \right]. \quad (13.47) \quad \{\mathbf{e:J2bound}\}$$

By plugging (13.46) and (13.47) into (13.33), we conclude

$$\begin{aligned} (13.33) &\lesssim \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G}, \tilde{\mathcal{G}} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) \left(\frac{1}{N^{3b/4}} \sum_{x \sim b_\alpha, x \neq a_\alpha} |G_{c_\alpha x}^{(b_\alpha)}|^2 + N^0 \Phi \right) \right] \\ &\lesssim \frac{1}{N^{3b/4} Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E} \left[\mathbf{1}(\mathcal{G} \in \Omega) I(\mathcal{F}^+, \mathcal{G}) \sum_{x \sim b_\alpha, x \neq a_\alpha} |G_{c_\alpha x}^{(b_\alpha)}|^2 \right] + \mathbb{E}[N^0 \Phi] \\ &\lesssim \frac{1}{N^{3b/4} (Nd)} \sum_{b_\alpha \sim c_\alpha} \mathbb{E} \left[I(\{c_\alpha, b_\alpha\}, \mathcal{G}) \sum_{x \sim b_\alpha, x \neq a_\alpha} |G_{c_\alpha x}^{(b_\alpha)}|^2 \right] + \mathbb{E}[N^0 \Phi] \\ &\lesssim \frac{1}{N^{1+3b/4}} \sum_{o \in [N]} \mathbb{E} \left[I(\{i, o\}, \mathcal{G}) |G_{ij}^{(o)}|^2 \right] + \mathbb{E}[N^0 \Phi], \end{aligned} \quad (13.48) \quad \{\mathbf{e:IIGU3}\}$$

where in the second statement, we dropped the indicator function $\mathbf{1}(\tilde{\mathcal{G}} \in \Omega)$; in the third statement sum over $\mathbf{i}^+ \setminus \{b_\alpha, c_\alpha\}$; for the last statement, we used the permutation invariance of the vertices, so that $G_{ij}^{(o)}$ and $G_{c_\alpha x}^{(b_\alpha)}$ have the same distribution.

Thus (13.33) and (13.43) together leads to the following bound

$$\frac{1}{N} \sum_{o \in \llbracket N \rrbracket} \mathbb{E}[I(\{i, o\}, \mathcal{G}) |G_{ij}^{(o)}|^2] \lesssim \frac{1}{N^{1+3b/4}} \sum_{o \in \llbracket N \rrbracket} \mathbb{E}[I(\{i, o\}, \mathcal{G}) |G_{ij}^{(o)}|^2] + \mathbb{E}[N^\circ \Phi],$$

and the claim (13.2) follows from rearranging. □

Proof of Proposition 13.2. Thanks to Lemma 13.3, we have

$$\tilde{G}_{c_\alpha c_\beta}^{(\mathbb{T})} = G_{c_\alpha c_\beta}^{(b_\alpha b_\beta)} + \mathcal{E}_{\alpha\beta},$$

where $\mathcal{E}_{\alpha\beta}$ as in (13.35). The statement follows from showing

$$\begin{aligned} \sum_{\gamma \in \llbracket \mu \rrbracket} \sum_{x \in \mathcal{N}_\gamma} \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})}|^2] &\lesssim (d-1)^\ell \mathbb{E}[N^\circ]. \\ \sum_{\gamma \in \llbracket \mu \rrbracket \setminus \{\alpha, \beta\}} \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha b_\gamma}^{(\mathbb{T}b_\alpha b_\beta)}|^2] &\lesssim (d-1)^\ell \mathbb{E}[N^\circ], \\ \sum_{x \in \mathbb{T}} \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha x}^{(b_\alpha b_\beta)}|^2] &\lesssim (d-1)^\ell \mathbb{E}[N^\circ], \end{aligned} \quad (13.49) \quad \{\mathbf{e:G2bound}\}$$

where $N_\gamma = \{x \neq c_\gamma : x \sim b_\gamma\} \cup \{a_\gamma\}$.

In the following we prove the first statement in (13.49), the other two are similar, so we omit their proofs. If $x \in \mathcal{N}_\gamma$ with $\gamma \neq \alpha$, or $x = a_\alpha$, we can first sum over $b_\alpha \sim c_\alpha$, and (3.27) gives

$$\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})}|^2 \lesssim N^\circ \Phi. \quad (13.50) \quad \{\mathbf{e:qinkuang1}\}$$

Otherwise $x \sim b_\alpha, x \neq c_\alpha$. We recall the upper bound on $|G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})} - G_{c_\alpha x}^{(b_\alpha)}|$ from (13.42) and (13.43)

$$\frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})} = \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) G_{c_\alpha x}^{(b_\alpha)} + O(N^{-b/2} \Phi). \quad (13.51) \quad \{\mathbf{e:qinkuang2}\}$$

By combining (13.50) and (13.51), we conclude

$$\begin{aligned} &\sum_{\gamma \in \llbracket \mu \rrbracket} \sum_{x \in \mathcal{N}_\gamma} \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha x}^{(\mathbb{T}\mathbb{W})}|^2] \\ &\lesssim \sum_{x \sim b_\alpha, x \neq c_\alpha} \frac{1}{Z_{\mathcal{F}^+}} \sum_{\mathbf{i}^+} \mathbb{E}[I(\mathcal{F}^+, \mathcal{G}) \mathbf{1}(\mathcal{G} \in \Omega) |G_{c_\alpha x}^{(b_\alpha)}|^2] + (d-1)^\ell \mathbb{E}[N^\circ \Phi] \lesssim (d-1)^\ell \mathbb{E}[N^\circ \Phi]. \end{aligned} \quad (13.52) \quad \{\mathbf{e:decomp}\}$$

where in the last line we used (13.2) to bound the first term; The first claim in (13.49) follows from combining (13.50) and (13.52). □

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